1. Monoidal categories

1.1. **The definition of a monoidal category.** A good way of thinking about category theory (which will be especially useful throughout these notes) is that category theory is a refinement (or "categorification") of ordinary algebra. In other words, there exists a dictionary between these two subjects, such that usual algebraic structures are recovered from the corresponding categorical structures by passing to the set of isomorphism classes of objects.

For example, the notion of a (small) category is a categorification of the notion of a set. Similarly, abelian categories are a categorification of abelian groups [1] (which justifies the terminology).

This dictionary goes surprisingly far, and many important constructions below will come from an attempt to enter into it a categorical "translation" of an algebraic notion.

In particular, the notion of a monoidal category is the categorification of the notion of a monoid.

Recall that a monoid may be defined as a set C with an associative multiplication operation $(x, y) \to x \cdot y$ (i.e., a semigroup), with an element 1 such that $1^2 = 1$ and the maps $1\cdot, \cdot 1 : C \to C$ are bijections. It is easy to show that in a semigroup, the last condition is equivalent to the usual unit axiom $1 \cdot x = x \cdot 1 = x$.

As usual in category theory, to categorify the definition of a monoid, we should replace the equalities in the definition of a monoid (namely, the associativity equation $(xy)z = x(yz)$ and the equation $1^2 = 1$) by isomorphisms satisfying some consistency properties, and the word "bijection" by the word "equivalence" (of categories). This leads to the following definition.

Definition 1.1.1. A monoidal category is a quintuple $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor called the *tensor product* bifunctor, $a : \bullet \otimes (\bullet \otimes \bullet) \xrightarrow{\sim} \bullet \otimes (\bullet \otimes \bullet)$ is a functorial isomorphism:

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \qquad X, Y, Z \in \mathcal{C} \tag{1.1.1}$$

called the *associativity constraint* (or *associativity isomorphism*), $\mathbf{1} \in \mathcal{C}$ is an object of $\mathcal{C}$, and $\iota : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ is an isomorphism, subject to the following two axioms.

[1] To be more precise, the set of isomorphism classes of objects in a (small) abelian category $\mathcal{C}$ is a commutative monoid, but one usually extends it to a group by considering "virtual objects" of the form $X - Y$, $X, Y \in \mathcal{C}$.

1. The pentagon axiom. The diagram

(1.1.2)

$$
\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W\otimes X,Y,Z}"'] \arrow[dr, "a_{W,X,Y}\otimes \mathrm{Id}_Z"] & \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y\otimes Z}"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W,X\otimes Y,Z}"] \\
W \otimes (X \otimes (Y \otimes Z)) & & W \otimes ((X \otimes Y) \otimes Z) \arrow[ll, "\mathrm{Id}_W \otimes a_{X,Y,Z}"']
\end{tikzcd}
$$

is commutative for all objects W, X, Y, Z in $\mathcal{C}$.

2. The unit axiom. The functors $L_{\mathbf{1}}$ and $R_{\mathbf{1}}$ of left and right multiplication by $\mathbf{1}$ are equivalences $\mathcal{C} \to \mathcal{C}$.

The pair $(\mathbf{1}, \iota)$ is called the *unit object* of $\mathcal{C}$.[2]

We see that the set of isomorphism classes of objects in a small monoidal category indeed has a natural structure of a monoid, with multiplication $\otimes$ and unit $\mathbf{1}$. Thus, in the categorical-algebraic dictionary, monoidal categories indeed correspond to monoids (which explains their name).

Definition 1.1.2. A monoidal subcategory of a monoidal category $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ is a quintuple $(\mathcal{D}, \otimes, a, \mathbf{1}, \iota)$, where $\mathcal{D} \subset \mathcal{C}$ is a subcategory closed under the tensor product of objects and morphisms and containing $\mathbf{1}$ and ι.

Definition 1.1.3. The *opposite monoidal category* $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}$ is the category $\mathcal{C}$ with reversed order of tensor product and inverted associativity somorphism.

Remark 1.1.4. The notion of the opposite monoidal category is not to be confused with the usual notion of the opposite category, which is the category $\mathcal{C}^{\vee}$ obtained from $\mathcal{C}$ by reversing arrows (for any category $\mathcal{C}$). Note that if $\mathcal{C}$ is monoidal, so is $\mathcal{C}^{\vee}$ (in a natural way), which makes it even easier to confuse the two notions.

1.2. **Basic properties of unit objects in monoidal categories.** Let $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ be a monoidal category. Define the isomorphism $l_X : \mathbf{1} \otimes X \to X$ by the formula

$$l_X = L_{\mathbf{1}}^{-1}((\iota \otimes \mathrm{Id}) \circ a_{\mathbf{1},\mathbf{1},X}^{-1}),$$

and the isomorphism $r_X : X \otimes \mathbf{1} \to X$ by the formula

$$r_X = R_{\mathbf{1}}^{-1}((\mathrm{Id} \otimes \iota) \circ a_{X,\mathbf{1},\mathbf{1}}).$$

---

[2]We note that there is no condition on the isomorphism ι, so it can be chosen arbitrarily.

This gives rise to functorial isomorphisms $l : L_{\mathbf{1}} \to \mathrm{Id}_{\mathcal{C}}$ and $r : R_{\mathbf{1}} \to \mathrm{Id}_{\mathcal{C}}$. These isomorphisms are called the *unit constraints* or *unit isomorphisms*. They provide the categorical counterpart of the unit axiom $1X = X1 = X$ of a monoid in the same sense as the associativity isomorphism provides the categorical counterpart of the associativity equation.

Proposition 1.2.1. *The "triangle" diagram*

(1.2.1)
$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, "a_{X,\mathbf{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{Id}_Y"'] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "\mathrm{Id}_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

is commutative for all $X, Y \in \mathcal{C}$. In particular, one has $r_{\mathbf{1}} = l_{\mathbf{1}} = \iota$.

Proof. This follows by applying the pentagon axiom for the quadruple of objects $X, \mathbf{1}, \mathbf{1}, Y$. More specifically, we have the following diagram:

(1.2.2)
$$\begin{tikzcd}
((X \otimes \mathbf{1}) \otimes \mathbf{1}) \otimes Y \arrow[rr, "a_{X,\mathbf{1},\mathbf{1}} \otimes \mathrm{Id}"] \arrow[dr, "r_X \otimes \mathrm{Id} \otimes \mathrm{Id}"'] \arrow[ddd, "a_{X \otimes \mathbf{1},\mathbf{1},Y}"'] & & (X \otimes (\mathbf{1} \otimes \mathbf{1})) \otimes Y \arrow[dl, "(\mathrm{Id} \otimes \iota) \otimes \mathrm{Id}"] \arrow[ddd, "a_{X,\mathbf{1} \otimes \mathbf{1},Y}"] \\
& (X \otimes \mathbf{1}) \otimes Y \arrow[d, "a_{X,\mathbf{1},Y}"'] & \\
& X \otimes (\mathbf{1} \otimes Y) & \\
(X \otimes \mathbf{1}) \otimes (\mathbf{1} \otimes Y) \arrow[ur, "r_X \otimes \mathrm{Id}"] \arrow[dr, "a_{X,\mathbf{1},\mathbf{1} \otimes Y}"'] & & X \otimes ((\mathbf{1} \otimes \mathbf{1}) \otimes Y) \arrow[ul, "\mathrm{Id} \otimes (\iota \otimes \mathrm{Id})"'] \arrow[dl, "\mathrm{Id} \otimes a_{\mathbf{1},\mathbf{1},Y}"] \\
& X \otimes (\mathbf{1} \otimes (\mathbf{1} \otimes Y)) \arrow[uu, "\mathrm{Id} \otimes l_{\mathbf{1} \otimes Y}"'] &
\end{tikzcd}$$

To prove the proposition, it suffices to establish the commutativity of the bottom left triangle (as any object of $\mathcal{C}$ is isomorphic to one of the form $\mathbf{1} \otimes Y$). Since the outside pentagon is commutative (by the pentagon axiom), it suffices to establish the commutativity of the other parts of the pentagon. Now, the two quadrangles are commutative due to the functoriality of the associativity isomorphisms, the commutativity of the upper triangle is the definition of r, and the commutativity of the lower right triangle is the definition of l.

The last statement is obtained by setting $X = Y = \mathbf{1}$ in (1.2.1). $\square$

Proposition 1.2.2. *The following diagrams commute for all objects $X, Y \in \mathcal{C}$:*

(1.2.3)
$$\begin{tikzcd}
(\mathbf{1} \otimes X) \otimes Y \arrow[rr, "a_{\mathbf{1},X,Y}"] \arrow[dr, "l_X \otimes \mathrm{Id}_Y"'] & & \mathbf{1} \otimes (X \otimes Y) \arrow[dl, "l_{X \otimes Y}"] \\
& X \otimes Y &
\end{tikzcd}$$

(1.2.4)
$$\begin{tikzcd}
(X \otimes Y) \otimes \mathbf{1} \arrow[rr, "a_{X,Y,\mathbf{1}}"] \arrow[dr, "r_{X \otimes Y}"'] & & X \otimes (Y \otimes \mathbf{1}) \arrow[dl, "\mathrm{Id}_X \otimes r_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

Proof. Consider the diagram

(1.2.5)
$$\begin{tikzcd}
((X \otimes \mathbf{1}) \otimes Y) \otimes Z \arrow[rr, "a_{X,\mathbf{1},Y} \otimes \mathrm{Id}"] \arrow[dr, "(r_X \otimes \mathrm{Id}) \otimes \mathrm{Id}"] \arrow[ddd, "a_{X \otimes \mathbf{1},Y,Z}"'] & & (X \otimes (\mathbf{1} \otimes Y)) \otimes Z \arrow[dl, "(\mathrm{Id} \otimes l_Y) \otimes \mathrm{Id}"'] \arrow[ddd, "a_{X,\mathbf{1} \otimes Y,Z}"] \\
& (X \otimes Y) \otimes Z \arrow[d, "a_{X,Y,Z}"'] & \\
& X \otimes (Y \otimes Z) & \\
(X \otimes \mathbf{1}) \otimes (Y \otimes Z) \arrow[ur, "r_X \otimes \mathrm{Id}"] \arrow[dr, "a_{X,\mathbf{1},Y \otimes Z}"'] & & X \otimes ((\mathbf{1} \otimes Y) \otimes Z) \arrow[ul, "\mathrm{Id} \otimes (l_Y \otimes \mathrm{Id})"'] \arrow[dl, "\mathrm{Id} \otimes a_{\mathbf{1},Y,Z}"] \\
& X \otimes (\mathbf{1} \otimes (Y \otimes Z)) \arrow[uu, "\mathrm{Id} \otimes l_{Y \otimes Z}"'] &
\end{tikzcd}$$

where X, Y, Z are objects in $\mathcal{C}$. The outside pentagon commutes by the pentagon axiom (1.1.2). The functoriality of a implies the commutativity of the two middle quadrangles. The triangle axiom (1.2.1) implies the commutativity of the upper triangle and the lower left triangle. Consequently, the lower right triangle commutes as well. Setting $X = \mathbf{1}$ and applying the functor $L_{\mathbf{1}}^{-1}$ to the lower right triangle, we obtain commutativity of the triangle (1.2.3). The commutativity of the triangle (1.2.4) is proved similarly. $\square$

Proposition 1.2.3. *For any object X in $\mathcal{C}$ one has the equalities $l_{\mathbf{1} \otimes X} = \mathrm{Id} \otimes l_X$ and $r_{X \otimes \mathbf{1}} = r_X \otimes \mathrm{Id}$.*

Proof. It follows from the functoriality of l that the following diagram commutes

$$\begin{tikzcd} \mathbf{1}\otimes(\mathbf{1}\otimes X) \arrow[r, "\mathrm{Id}\otimes l_X"] \arrow[d, "l_{\mathbf{1}\otimes X}"'] & \mathbf{1}\otimes X \arrow[d, "l_X"] \\ \mathbf{1}\otimes X \arrow[r, "l_X"'] & X \end{tikzcd} \tag{1.2.6}$$

Since l_X is an isomorphism, the first identity follows. The second identity follows similarly from the functoriality of r. $\square$

Proposition 1.2.4. *The unit object in a monoidal category is unique up to a unique isomorphism.*

Proof. Let $(\mathbf{1},\iota),(\mathbf{1}',\iota')$ be two unit objects. Let (r,l), (r',l') be the corresponding unit constraints. Then we have the isomorphism $\eta := l_{\mathbf{1}'}\circ(r'_{\mathbf{1}})^{-1} : \mathbf{1}\to\mathbf{1}'$.

It is easy to show using commutativity of the above triangle diagrams that η maps ι to ι'. It remains to show that η is the only isomorphism with this property. To do so, it suffices to show that if $b:\mathbf{1}\to\mathbf{1}$ is an isomorphism such that the diagram

$$\begin{tikzcd} \mathbf{1}\otimes\mathbf{1} \arrow[r, "b\otimes b"] \arrow[d, "\iota"'] & \mathbf{1}\otimes\mathbf{1} \arrow[d, "\iota"] \\ \mathbf{1} \arrow[r, "b"'] & \mathbf{1} \end{tikzcd} \tag{1.2.7}$$

is commutative, then $b=\mathrm{Id}$. To see this, it suffices to note that for any morphism $c:\mathbf{1}\to\mathbf{1}$ the diagram

$$\begin{tikzcd} \mathbf{1}\otimes\mathbf{1} \arrow[r, "c\otimes \mathrm{Id}"] \arrow[d, "\iota"'] & \mathbf{1}\otimes\mathbf{1} \arrow[d, "\iota"] \\ \mathbf{1} \arrow[r, "c"'] & \mathbf{1} \end{tikzcd} \tag{1.2.8}$$

is commutative (as $\iota = r_{\mathbf{1}}$), so $b\otimes b = b\otimes\mathrm{Id}$ and hence $b=\mathrm{Id}$. $\square$

Exercise 1.2.5. Verify the assertion in the proof of Proposition 1.2.4 that η maps ι to ι'.

Hint. Use Propositions 1.2.1 and 1.2.2.

The results of this subsection show that a monoidal category can be alternatively defined as follows:

Definition 1.2.6. A monoidal category is a sextuple $(\mathcal{C},\otimes,a,\mathbf{1},l,r)$ satisfying the pentagon axiom (1.1.2) and the triangle axiom (1.2.1).

This definition is perhaps more traditional than Definition 1.1.1, but Definition 1.1.1 is simpler. Besides, Proposition 1.2.4 implies that for a triple $(\mathcal{C}, \otimes, a)$ satisfying a pentagon axiom (which should perhaps be called a "semigroup category", as it categorifies the notion of a semigroup), being a monoidal category is a property and not a structure (similarly to how it is for semigroups and monoids).

Furthermore, one can show that the commutativity of the triangles implies that in a monoidal category one can safely identify $\mathbf{1} \otimes X$ and $X \otimes \mathbf{1}$ with X using the unit isomorphisms, and assume that the unit isomorphism are the identities (which we will usually do from now on).[3]

In a sense, all this means that in constructions with monoidal categories, unit objects and isomorphisms always "go for the ride", and one need not worry about them especially seriously. For this reason, below we will typically take less care dealing with them than we have done in this subsection.

Proposition 1.2.7. *(*[SR, 1.3.3.1]*) The monoid* $\mathsf{End}(\mathbf{1})$ *of endomorphisms of the unit object of a monoidal category is commutative.*

Proof. The unit isomorphism $\iota : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ induces the isomorphism $\psi : \mathsf{End}(\mathbf{1} \otimes \mathbf{1}) \xrightarrow{\sim} \mathsf{End}(\mathbf{1})$. It is easy to see that $\psi(a \otimes 1) = \psi(1 \otimes a) = a$ for any $a \in \mathsf{End}(\mathbf{1})$. Therefore,

$$ab = \psi((a \otimes 1)(1 \otimes b)) = \psi((1 \otimes b)(a \otimes 1)) = ba, \tag{1.2.9}$$

for any $a, b \in \mathsf{End}(\mathbf{1})$. □

1.3. **First examples of monoidal categories.** Monoidal categories are ubiquitous. You will see one whichever way you look. Here are some examples.

Example 1.3.1. The category **Sets** of sets is a monoidal category, where the tensor product is the Cartesian product and the unit object is a one element set; the structure morphisms a, ι, l, r are obvious. The same holds for the subcategory of finite sets, which will be denoted by Sets [4]. This example can be widely generalized: one can take the category of sets with some structure, such as groups, topological spaces, etc.

Example 1.3.2. Any additive category is monoidal, with $\otimes$ being the direct sum functor $\oplus$, and $\mathbf{1}$ being the zero object.

The remaining examples will be especially important below.

[3]We will return to this issue later when we discuss MacLane's coherence theorem.

[4]Here and below, the absence of a finiteness condition condition is indicated by the boldface font, while its presence is indicated by the Roman font.

Example 1.3.3. Let k be any field. The category $k - \mathbf{Vec}$ of all $k-$vector spaces is a monoidal category, where $\otimes = \otimes_k$, $\mathbf{1} = k$, and the morphisms a, ι, l, r are the obvious ones. The same is true about the category of finite dimensional vector spaces over k, denoted by $k - \mathrm{Vec}$. We will often drop k from the notation when no confusion is possible.

More generally, if R is a commutative unital ring, then replacing k by R we can define monoidal categories $R - \mathbf{mod}$ of R-modules and $R - \mathrm{mod}$ of R-modules of finite type.

Example 1.3.4. Let G be a group. The category $\mathbf{Rep}_k(G)$ of all representations of G over k is a monoidal category, with $\otimes$ being the tensor product of representations: if for a representation V one denotes by ρ_V the corresponding map $G \to GL(V)$, then

$$\rho_{V\otimes W}(g) := \rho_V(g) \otimes \rho_W(g).$$

The unit object in this category is the trivial representation $\mathbf{1} = k$. A similar statement holds for the category $\mathrm{Rep}_k(G)$ of finite dimensional representations of G. Again, we will drop the subscript k when no confusion is possible.

Example 1.3.5. Let G be an affine (pro)algebraic group over k. The categories $\mathrm{Rep}(G)$ of all algebraic representations of G over k is a monoidal category (similarly to Example 1.3.4).

Similarly, if $\mathfrak{g}$ is a Lie algebra over k, then the category of its representations $\mathbf{Rep}(\mathfrak{g})$ and the category of its finite dimensional representations $\mathrm{Rep}(\mathfrak{g})$ are monoidal categories: the tensor product is defined by

$$\rho_{V\otimes W}(a) = \rho_V(a) \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes \rho_W(a)$$

(where $\rho_Y : \mathfrak{g} \to \mathfrak{gl}(Y)$ is the homomorphism associated to a representation Y of $\mathfrak{g}$), and $\mathbf{1}$ is the 1-dimensional representation with the zero action of $\mathfrak{g}$.

Example 1.3.6. Let G be a monoid (which we will usually take to be a group), and let A be an abelian group (with operation written multiplicatively). Let $\mathcal{C}_G = \mathcal{C}_G(A)$ be the category whose objects δ_g are labeled by elements of G (so there is only one object in each isomorphism class), $\mathsf{Hom}(\delta_{g_1}, \delta_{g_2}) = \emptyset$ if $g_1 \neq g_2$, and $\mathsf{Hom}(\delta_g, \delta_g) = A$, with the functor $\otimes$ defined by $\delta_g \otimes \delta_h = \delta_{gh}$, and the tensor tensor product of morphisms defined by $a \otimes b = ab$. Then $\mathcal{C}_G$ is a monoidal category with the associativity isomorphism being the identity, and $\mathbf{1}$ being the unit element of G. This shows that in a monoidal category, $X \otimes Y$ need not be isomorphic to $Y \otimes X$ (indeed, it suffices to take a non-commutative monoid G).

This example has a "linear" version. Namely, let k be a field, and $k - \mathbf{Vec}_G$ denote the category of G-graded vector spaces over k, i.e. vector spaces V with a decomposition $V = \oplus_{g\in G} V_g$. Morphisms in this category are linear operators which preserve the grading. Define the tensor product on this category by the formula

$$(V \otimes W)_g = \oplus_{x,y\in G: xy=g} V_x \otimes W_y,$$

and the unit object $\mathbf{1}$ by $\mathbf{1}_1 = k$ and $\mathbf{1}_g = 0$ for $g \neq 1$. Then, defining a, ι in an obvious way, we equip $k - \mathbf{Vec}_G$ with the structure of a monoidal category. Similarly one defines the monoidal category $k - \mathrm{Vec}_G$ of finite dimensional G-graded k-vector spaces.

In the category $k - \mathrm{Vec}_G$, we have pairwise non-isomorphic objects δ_g, $g \in G$, defined by the formula $(\delta_g)_x = k$ if $x = g$ and $(\delta_g)_x = 0$ otherwise. For these objects, we have $\delta_g \otimes \delta_h \cong \delta_{gh}$. Thus the category $\mathcal{C}_G(k^\times)$ is a (non-full) monoidal subcategory of $k - \mathrm{Vec}_G$. This subcategory can be viewed as a "basis" of Vec_G (and Vec_G as "the linear span" of $\mathcal{C}_G$), as any object of Vec_G is isomorphic to a direct sum of objects δ_g with nonnegative integer multiplicities.

When no confusion is possible, we will denote the categories $k - \mathrm{Vec}_G$, $k - \mathbf{Vec}_G$ simply by Vec_G, $\mathbf{Vec}_G$.

Example 1.3.7. This is really a generalization of Example 1.3.6, which shows that the associativity isomorphism is not always "the obvious one".

Let G be a group, A an abelian group, and ω be a 3-cocycle of G with values in A. This means that $\omega : G \times G \times G \to A$ is a function satisfying the equation

(1.3.1)

$$\omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2g_3, g_4)\omega(g_2, g_3, g_4),$$

for all $g_1, g_2, g_3, g_4 \in G$.

Let us define the monoidal category $\mathcal{C}_G^\omega = \mathcal{C}_G^\omega(A)$ as follows. As a category, it is the same as the category $\mathcal{C}_G$ defined above. The bifunctor $\otimes$ and the unit object $(\mathbf{1}, \iota)$ in this category is also the same as those in $\mathcal{C}_G$. The only difference is in the new associativity isomorphism a^ω, which is not "the obvious one" (i.e., the identity) like in $\mathcal{C}_G$, but rather is defined by the formula

$$a^\omega_{\delta_g,\delta_h,\delta_m} = \omega(g, h, m) : (\delta_g \otimes \delta_h) \otimes \delta_m \to \delta_g \otimes (\delta_h \otimes \delta_m), \tag{1.3.2}$$

where $g, h, m \in G$.

The fact that $\mathcal{C}_G^\omega$ with these structures is indeed a monoidal category follows from the properties of ω. Namely, the pentagon axiom (1.1.2) follows from equation (1.3.1), and the unit axiom is obvious.

Similarly, for a field k, one can define the category $(k-)\mathbf{Vec}_G^\omega$, which differs from $\mathbf{Vec}_G$ just by the associativity isomorphism. This is done by extending the associativity isomorphism of $\mathcal{C}_G^\omega$ by additivity to arbitrary direct sums of objects δ_g. This category contains a monoidal subcategory Vec_G^ω of finite dimensional G-graded vector spaces with associativity defined by ω.

Remark 1.3.8. It is straightforward to verify that the unit morphisms l, r in $\mathbf{Vec}_G^\omega$ are given on 1-dimensional spaces by the formulas

$$l_{\delta_g} = \omega(1,1,g)^{-1}\mathrm{Id}_g,\ \ r_{\delta_g} = \omega(g,1,1)\mathrm{Id}_g,$$

and the triangle axiom says that $\omega(g,1,h) = \omega(g,1,1)\omega(1,1,h)$. Thus, we have $l_X = r_X = \mathrm{Id}$ if and only if

$$\omega(g,1,1) = \omega(1,1,g), \tag{1.3.3}$$

for any $g \in G$ or, equivalently,

$$\omega(g,1,h) = 1,\ \ g,h \in G. \tag{1.3.4}$$

A cocycle satisfying this condition is said to be *normalized*.

Example 1.3.9. Let $\mathcal{C}$ be a category. Then the category $\mathsf{End}(\mathcal{C})$ of all functors from $\mathcal{C}$ to itself is a monoidal category, where $\otimes$ is given by composition of functors. The associativity isomorphism in this category is the identity. The unit object is the identity functor, and the structure morphisms are obvious. If $\mathcal{C}$ is an abelian category, the same is true about the categories of additive, left exact, right exact, and exact endofunctors of $\mathcal{C}$.

Example 1.3.10. Let A be an associative ring with unit. Then the category $A-\mathbf{bimod}$ of bimodules over A is a monoidal category, with tensor product $\otimes = \otimes_A$, over A. The unit object in this category is the ring A itself (regarded as an A-bimodule).

If A is commutative, this category has a full monoidal subcategory $A-\mathbf{mod}$, consisting of A-modules, regarded as bimodules in which the left and right actions of A coincide. More generally, if X is a scheme, one can define the monoidal category $\mathbf{QCoh}(X)$ of quasicoherent sheaves on X; if X is affine and $A = \mathcal{O}_X$, then $\mathbf{QCoh}(X) = A-\mathbf{mod}$.

Similarly, if A is a finite dimensional algebra, we can define the monoidal category $A-$bimod of finite dimensional A-bimodules. Other similar examples which often arise in geometry are the category $\mathrm{Coh}(X)$ of coherent sheaves on a scheme X, its subcategory $\mathrm{VB}(X)$ of vector bundles (i.e., locally free coherent sheaves) on X, and the category $\mathrm{Loc}(X)$ of locally constant sheaves of finite dimensional k-vector spaces (also called local systems) on any topological space X. All of these are monoidal categories in a natural way.

Example 1.3.11. The category of tangles.

Let $S_{m,n}$ be the disjoint union of m circles $\mathbb{R}/\mathbb{Z}$ and n intervals $[0,1]$. A *tangle* is a piecewise smooth embedding $f : S_{m,n} \to \mathbb{R}^2 \times [0,1]$ such that the boundary maps to the boundary and the interior to the interior. We will abuse the terminology by also using the term "tangle" for the image of f.

Let x, y, z be the Cartesian coordinates on $\mathbb{R}^2 \times [0,1]$. Any tangle has inputs (points of the image of f with $z = 0$) and outputs (points of the image of f with $z = 1$). For any integers $p, q \geq 0$, let $\widetilde{T}_{p,q}$ be the set of all tangles which have p inputs and q outputs, all having a vanishing y-coordinate. Let $T_{p,q}$ be the set of isotopy classes of elements of $\widetilde{T}_{p,q}$; thus, during an isotopy, the inputs and outputs are allowed to move (preserving the condition $y = 0$), but cannot meet each other. We can define a canonical composition map $T_{p,q} \times T_{q,r} \to T_{p,r}$, induced by the concatenation of tangles. Namely, if $s \in T_{p,q}$ and $t \in T_{q,r}$, we pick representatives $\widetilde{s} \in \widetilde{T}_{p,q}, \widetilde{t} \in \widetilde{T}_{q,r}$ such that the inputs of $\widetilde{t}$ coincide with the outputs of $\widetilde{s}$, concatenate them, perform an appropriate reparametrization, and rescale $z \to z/2$. The obtained tangle represents the desired composition ts.

We will now define a monoidal category $\mathcal{T}$ called the category of tangles (see [K, T, BaKi] for more details). The objects of this category are nonnegative integers, and the morphisms are defined by $\mathsf{Hom}_{\mathcal{T}}(p,q) = T_{p,q}$, with composition as above. The identity morphisms are the elements $\mathrm{id}_p \in T_{p,p}$ represented by p vertical intervals and no circles (in particular, if $p = 0$, the identity morphism id_p is the empty tangle).

Now let us define the monoidal structure on the category $\mathcal{T}$. The tensor product of objects is defined by $m \otimes n = m + n$. However, we also need to define the tensor product of morphisms. This tensor product is induced by union of tangles. Namely, if $t_1 \in T_{p_1,q_1}$ and $t_2 \in T_{p_2,q_2}$, we pick representatives $\widetilde{t_1} \in \widetilde{T}_{p_1,q_1}, \widetilde{t_2} \in \widetilde{T}_{p_2,q_2}$ in such a way that any point of $\widetilde{t_1}$ is to the left of any point of $\widetilde{t_2}$ (i.e., has a smaller x-coordinate). Then $t_1 \otimes t_2$ is represented by the tangle $\widetilde{t_1} \cup \widetilde{t_2}$.

We leave it to the reader to check the following:

1. The product $t_1 \otimes t_2$ is well defined, and its definition makes $\otimes$ a bifunctor.

2. There is an obvious associativity isomorphism for $\otimes$, which turns $\mathcal{T}$ into a monoidal category (with unit object being the empty tangle).