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# A Centennial: Wilhelm Killing and the Exceptional Groups

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In his article [3] in the *Mathematical Intelligencer* (vol. 11, no. 3) A. John Coleman gives a colorful biography of the mathematician Wilhelm Killing and offers an admiring appraisal of his paper [8]. The subject of this paper, the classification of the simple Lie algebras over  $\mathbb{C}$ , has indeed turned out to be a milestone in the history of mathematics. At the conclusion of his article Coleman lists six reasons why he considers [8] to be such an epoch-making paper, the first one being that it furnished the impetus towards the problem of classifying finite simple groups. Here one could add that the answer to that problem was also motivated by and was partly provided by the classification of simple Lie algebras through Claude Chevalley's paper [2b].

In two sections of his article, entitled "Killing Intervenes" and "The Still Point of the Turning World," Coleman discusses the work of Killing and Élie Cartan on the classification of simple Lie algebras over  $\mathbb{C}$ . I would like to add a few comments to his discussion (see also [7b]).

While Sophus Lie and some of his associates in Leipzig attempted the problem of classifying all local transformation groups of  $\mathbb{R}^n$ , Killing set himself the problem of finding all possible *Zusammensetzungen* of  $r$ -parameter groups. In other words, he was interested in all possible ways in which a vector space could be turned into a Lie algebra. While Lie was motivated by applications to differential equations, Killing was led to his problem from his work in geometry.

Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$ . Then  $\mathfrak{g}$  is isomorphic to the Lie algebra of linear transformations  $\text{ad}X$  of  $\mathfrak{g}$  given by  $\text{ad}X(Y) = [X, Y]$ , with  $X$  running through  $\mathfrak{g}$  and  $[, ]$  denoting the bracket product in the Lie algebra. To study this family it is natural to try to diagonalize the operators  $\text{ad}(X)$  as effectively as possible. This is the motivation for the definition of a Cartan subalgebra as a subalgebra  $\mathfrak{h}$  which is

- (i) A maximal abelian subalgebra of  $\mathfrak{g}$ .
- (ii) For each  $H \in \mathfrak{h}$ ,  $\text{ad}H$  is a diagonalizable linear transformation of  $\mathfrak{g}$ .

For the classical simple Lie algebras the existence of such an  $\mathfrak{h}$  is a simple matter; for the general case Killing gave only an incomplete proof. The gap was filled in Cartan's thesis. Even today the general existence proof for a Cartan subalgebra is not easy. The usual proof proceeds using theorems of Engel and of Lie on nilpotent and solvable Lie algebras, respectively; an entirely different proof, realizing a possibility suggested by Cartan [1e], p. 23, was devised by Roger Richardson [11]. In [8] Killing introduced many of the fundamental concepts in the modern theory of simple Lie algebras, for example the following:

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(i) The roots of  $\mathfrak{g}$ , which are by his definition the roots of the characteristic equation

$$\det(\lambda I - \text{ad } X) = 0.$$

Twice the second coefficient in this equation, which equals  $\text{Tr}(\text{ad } X)^2$ , is now customarily called the *Killing form*. However, Cartan made much more use of this form.

(ii) A basis  $\alpha_1, \dots, \alpha_l$  of the roots; all other roots are integral linear combinations. He then introduced the matrix  $(a_{ij})$  where

$$-a_{ij} = \text{the largest integer } q \text{ such that } \alpha_i + q\alpha_j \text{ is a root.}$$

The matrix  $(a_{ij})$  is now called the *Cartan matrix*.

Killing had discovered that the bracket relations in  $\mathfrak{g}$ , in particular the Jacobi identity, are reflected in certain additive properties of the roots. This motivated his introduction of  $(a_{ij})$  described above. Killing's method of classification is a grandiose conception and consists of two main steps:

Step I. Find certain necessary conditions on the matrix  $(a_{ij})$  (cf. Killing [8], §13 or Cartan [1a], § 5). Then classify all equivalence classes of matrices  $(a_{ij})$  satisfying these conditions.

Step II. Show that for each equivalence class of matrices  $(a_{ij})$  there exists exactly one simple Lie algebra over  $\mathbb{C}$  having it as a Cartan matrix.

Step II is explicitly stated in Cartan [1a], beginning of § 8.

Roughly speaking, this method is one that is still used today (among others). Instead of Step I, one carries out the equivalent classification of root systems by means of the Coxeter-Dynkin diagrams.

As stated by Cartan [1a] p. 410, Step I above was completely carried out by Killing. While Killing recognized that  $A_3 = D_3$  (the local isomorphism between  $SU(4)$  and  $SO(6)$ ) he did not notice that  $E_4 = F_4$ , although as Cartan remarked this is immediate from his root tables in [8], pp. 30–31.

While  $\text{Trace}(\text{ad } X)^2$  is nowadays called the Killing form and the matrix  $(a_{ij})$  called the Cartan matrix, in view of the above it would have been reasonable on historical grounds to interchange the names.

Step II above, the existence and uniqueness of a simple Lie algebra with a given Cartan matrix, is a more difficult problem. This is where Killing's work is most defective, although the existence result at the end of [8] is correct. For the matrices  $(a_{ij})$  for the classical Lie algebras  $A_l, B_l, C_l, D_l$  the uniqueness is stated in Killing ([8], p. 42) and is verified in detail in Cartan [1b], Ch. V, p. 72–87.

The exceptional simple Lie algebras are the subject



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of the final §18 in Killing's paper. This is certainly his most remarkable discovery, although these algebras appeared to him at first as a kind of a nuisance, which he tried hard to eliminate. Even Lie, who was generally critical of Killing's work, expressed in letters to Felix Klein his admiration of the results [6a]. While these exceptional Lie algebras may at first have been rather unwelcome, they have subsequently played important roles in Lie theory, for example by forcing one to find *a priori* conceptual proofs rather than case-by-case verifications. Extrapolation from the classical Lie algebras has to be done with care: for example in [7c] a result appears about invariants, which holds for all the classical Lie algebras (real or complex) but fails for exactly four of the seventeen real exceptional Lie algebras. It may at first appear strange that after working so hard at obtaining the classification, mathematicians should work equally hard avoiding the use thereof. The aim is of course thereby to gain better understanding. The role of the exceptional Lie groups in the construction of the sporadic simple finite groups and the use of the exceptional groups in modern string theory are among the many unexpected bonuses.

For the actual existence of the exceptional Lie algebras Killing indicates in §18 how to determine the structural constants on the basis of the matrix  $(a_{ij})$ ; the Jacobi identity then has to be verified in each case. Killing does this for  $G_2$ ; for the others this would necessitate a huge computation and from the indications in Killing ([8], p. 48), it is hard to say how far he had progressed. Killing had also tried to represent  $G_2$  as a

genuine local transformation group in  $\mathbb{R}^n$ . He found that  $n$  would necessarily have to be larger than 4; when he communicated this to Friedrich Engel, he (Engel [4a]) and independently Cartan ([1b], p. 281) showed that  $G_2$  could be realized as the stability group of the system

$$dx_3 + x_1 dx_2 - x_2 dx_1 = 0$$

$$dx_4 + x_3 dx_1 - x_1 dx_3 = 0$$

$$dx_5 + x_2 dx_3 - x_3 dx_2 = 0$$

in  $\mathbb{R}^5$ .

In his papers [1a], [1b] Cartan devotes much work to Step II (i.e., existence and uniqueness) for the excep-

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tional groups. In [1a] he states without proof an explicit representation for  $F_4$  as a stability group of a Pfaffian system in  $\mathbb{R}^{15}$  (in analogy with  $G_2$  above) and indicates representations for  $E_6$ ,  $E_7$ , and  $E_8$  as groups of contact transformations in  $\mathbb{R}^{16}$ ,  $\mathbb{R}^{27}$ , and  $\mathbb{R}^{29}$ , respectively. In his thesis Cartan determines the structural constants on the basis of the matrix  $(a_{ij})$ , remarking explicitly how this implies the uniqueness (cf. [1b, p. 93]). For the existence, the Jacobi identity would have to be verified; since the structural constants given by Cartan have a very simple and symmetric form it is possible that he indeed did verify the Jacobi identity, but he is silent on this point. This verification would be unnecessary if the above models that Cartan gives in [1a] could be shown to have Lie algebras with the structural constants mentioned. In his thesis he also gives a second set of models for the exceptional groups, but again with somewhat sketchy proofs. Nevertheless, one is completely convinced that Cartan proved for himself both the existence and uniqueness of the exceptional groups and that the full details were left out only because of their complexity.

Subsequently, general *a priori* proofs have been given for both parts of Step II. The uniqueness was proved by Hermann Weyl [12]; for the existence, a proof was given by Ernst Witt [13] and by Chevalley [2a] (with full details in Harish-Chandra [5]).

Coming back to Coleman's article, I agree with him that in the past Killing's work has been overshadowed by the work of Cartan; however, in recent years I believe that Killing's work has been better recognized and Coleman's article is also a valuable contribution in this regard.

Coleman also asks: Why was Killing's work neglected? I think that in modern terminology it is fair to

say that Cartan's thesis represented a friendly take-over. In addition to many other new results, Cartan's thesis gave a complete solution of the classification problem, so at least for this problem it was no longer necessary to struggle with Killing's unrigorous papers. In addition to giving a clear exposition, Cartan's thesis represented first-rate scholarship; in particular it gives precise references to and a fair and detailed evaluation of Killing's work, warts and all. Cartan's thesis is actually much easier reading than his later papers in differential geometry, which reflected his accumulated experience with Lie groups combined with his unusual geometric intuition.

A more challenging question seems to be: Why did it take so long for Cartan's thesis to get assimilated in spite of the recognized importance of its results and the clarity of its exposition? Although the results seem specific and concrete to us, perhaps Cartan's contemporaries saw them as relatively abstract, in that application of specific transformation groups to differential equations was no longer the dominant theme. At any rate, it seems that Cartan had the field of Lie algebra theory pretty much to himself for about a quarter century. (A notable exception: the Levi decomposition in paper [9] from 1905). Here it should perhaps be noted that Cartan's Lie algebra papers subsequent to his thesis, for example his paper [1d] on the classification of simple Lie algebras over  $\mathbb{R}$  and his paper [1c] on representation theory, were considerably more difficult than his thesis.

For the structure theory and classification of simple Lie algebras  $\mathfrak{g}$  over  $\mathbb{R}$ , the basic tool is the concept of a Cartan subspace  $\alpha$  defined by the properties:

- (i) For each  $H \in \alpha$ ,  $\text{ad}H$  is a (real) diagonalizable linear transformation of  $\mathfrak{g}$ .
- (ii)  $\alpha$  is maximal relative to (i).

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The corresponding root theory is more complicated in that twice a root can now be a root and, in addition, the multiplicity of a root may now be larger than one. From the classification one can deduce that the root pattern, together with the multiplicities, determines  $\mathfrak{g}$  up to isomorphism ([7a] p. 535); however, a simple direct proof of this fact does not seem to be available.

On page 30 of [3] Coleman speculates that perhaps Cartan might not have become a research mathematician if he had been unaware of Killing's work (for example, if it had never been published for lack of rigor). With my admiration of Cartan's genius, I consider this possibility extremely remote. Cartan was at the top of

his class at École Normale [6b]; he and fellow students had been very much immersed in the work of Lie as well as in differential geometry through the teaching of Gaston Darboux. If Killing had not come along, Cartan might have worked on infinite Lie groups earlier than he did (in 1902) as well as on differential systems (1899) and differential geometry (1910). If not earlier, it seems quite possible that he would have discovered the classification of simple Lie algebras through his differential geometric work on symmetric

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spaces (1926), because their classification is equivalent to that of the simple Lie algebras over  $\mathbf{R}$ . This is of course pure speculation.

It seems ironic that while Lie [10, vol. 3, pp. 768–771] subjected Killing's work to a severe criticism, one can argue that Killing's paper [8] was the first spark that led to the forging of the theory of Lie groups and Lie algebras into a mathematical force in its own right, independent of differential equations, exerting ever-increasing influence in mathematics and mathematical physics. Therefore it seems appropriate for all mathematicians to commemorate Wilhelm Killing on this centenary of his epoch-making paper.

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