3. The rational Cherednik algebra

3.1. **Definition and examples.** Above we have made essential use of the commutation relations between operators $x \in \mathfrak{h}^*, g \in G$, and $D_a, a \in \mathfrak{h}$. This makes it natural to consider the algebra generated by these operators.

Definition 3.1. The rational Cherednik algebra associated to $(G, \mathfrak{h})$ is the algebra $H_c(G, \mathfrak{h})$ generated inside $A = \mathrm{Rees}(\mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}))$ by the elements $x \in \mathfrak{h}^*, g \in G$, and $D_a(c, \hbar), a \in \mathfrak{h}$. If $t \in \mathbb{C}$, then the algebra $H_{t,c}(G, \mathfrak{h})$ is the specialization of $H_c(G, \mathfrak{h})$ at $\hbar = t$.

Proposition 3.2. *The algebra H_c is the quotient of the algebra $\mathbb{C}G \ltimes \mathbf{T}(\mathfrak{h} \oplus \mathfrak{h}^*)[\hbar]$ (where $\mathbf{T}$ denotes the tensor algebra) by the ideal generated by the relations*

$$[x, x'] = 0, \ [y, y'] = 0, \ [y, x] = \hbar(y, x) - \sum_{s \in \mathcal{S}} c_s(y, \alpha_s)(x, \alpha_s^\vee)s,$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$.

Proof. Let us denote the algebra defined in the proposition by $H'_c = H'_c(G, \mathfrak{h})$. Then according to the results of the previous sections, we have a surjective homomorphism $\phi : H'_c \to H_c$ defined by the formula $\phi(x) = x$, $\phi(g) = g$, $\phi(y) = D_y(c, \hbar)$.

Let us show that this homomorphism is injective. For this purpose assume that y_i is a basis of $\mathfrak{h}$, and x_i is the dual basis of $\mathfrak{h}^*$. Then it is clear from the relations of H'_c that H'_c is spanned over $\mathbb{C}[\hbar]$ by the elements

$$g \prod_{i=1}^{r} y_i^{m_i} \prod_{i=1}^{r} x_i^{n_i}. \tag{3.1}$$

Thus it remains to show that the images of the elements (3.1) under the map ϕ, i.e. the elements

$$g \prod_{i=1}^{r} D_{y_i}(c, \hbar)^{m_i} \prod_{i=1}^{r} x_i^{n_i}.$$

are linearly independent. But this follows from the obvious fact that the symbols of these elements in $\mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{\mathrm{reg}}][\hbar]$ are linearly independent. The proposition is proved. ☐

Remark 3.3. 1. Similarly, one can define the universal algebra $H(G, \mathfrak{h})$, in which both $\hbar$ and c are variables. (So this is an algebra over $\mathbb{C}[\hbar, c]$.) It has two equivalent definitions similar to the above.

2. It is more convenient to work with algebras defined by generators and relations than with subalgebras of a given algebra generated by a given set of elements. Therefore, from now on we will use the statement of Proposition 3.2 as a definition of the rational Cherednik algebra H_c. According to Proposition 3.2, this algebra comes with a natural embedding $\Theta_c : H_c \to \mathrm{Rees}(\mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}))$, defined by the formula $x \to x$, $g \to g$, $y \to D_y(c, \hbar)$. This embedding is called *the Dunkl operator embedding*.

Example 3.4. 1. Let $G = \mathbb{Z}_2$, $\mathfrak{h} = \mathbb{C}$. In this case c reduces to one parameter, and the algebra $H_{t,c}$ is generated by elements x, y, s with defining relations

$$s^2 = 1, \ sx = -xs, \ sy = -ys, \ [y, x] = t - 2cs.$$

2. Let $G = \mathfrak{S}_n$, $\mathfrak{h} = \mathbb{C}^n$. In this case there is also only one complex parameter c, and the algebra $H_{t,c}$ is the quotient of $\mathfrak{S}_n \ltimes \mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$ by the relations

$$[x_i, x_j] = [y_i, y_j] = 0,\ [y_i, x_j] = cs_{ij},\ [y_i, x_i] = t - c\sum_{j\neq i} s_{ij}.$$

Here $\mathbb{C}\langle E\rangle$ denotes the free algebra on a set E, and s_{ij} is the transposition of i and j.

3.2. **The PBW theorem for the rational Cherednik algebra.** Let us put a filtration on H_c by setting $\deg y = 1$ for $y \in \mathfrak{h}$ and $\deg x = \deg g = 0$ for $x \in \mathfrak{h}^*, g \in G$. Let $\mathrm{gr}(H_c)$ denote the associated graded algebra of H_c under this filtration, and similarly for $H_{t,c}$. We have a natural surjective homomorphism

$$\xi : \mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*][\hbar] \to \mathrm{gr}(H_c).$$

For $t \in \mathbb{C}$, it specializes to surjective homomorphisms

$$\xi_t : \mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \to \mathrm{gr}(H_{t,c}).$$

Proposition 3.5 (The PBW theorem for rational Cherednik algebras)**.** *The maps ξ and ξ_t are isomorphisms.*

Proof. The statement is equivalent to the claim that the elements (3.1) are a basis of $H_{t,c}$, which follows from the proof of Proposition 3.2. □

Remark 3.6. 1. We have

$$H_{0,0} = \mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \text{ and } H_{1,0} = \mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}).$$

2. For any $\lambda \in \mathbb{C}^*$, the algebra $H_{t,c}$ is naturally isomorphic to $H_{\lambda t,\lambda c}$.

3. The Dunkl operator embedding Θ_c specializes to embeddings

$$\Theta_{0,c} : H_{0,c} \hookrightarrow \mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{\mathrm{reg}}],$$

given by $x \mapsto x$, $g \mapsto g$, $y \mapsto D_a^0$, and

$$\Theta_{1,c} : H_{1,c} \hookrightarrow \mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}),$$

given by $x \mapsto x$, $g \mapsto g$, $y \mapsto D_a$. So $H_{0,c}$ is generated by x, g, D_a^0, and $H_{1,c}$ is generated by x, g, D_a.

Since Dunkl operators map polynomials to polynomials, the map $\Theta_{1,c}$ defines a representation of $H_{1,c}$ on $\mathbb{C}[\mathfrak{h}]$. This representation is called the *polynomial representation* of $H_{1,c}$.

3.3. **The spherical subalgebra.** Let $\mathsf{e} \in \mathbb{C}G$ be the symmetrizer, $\mathsf{e} = |G|^{-1}\sum_{g\in G} g$. We have $\mathsf{e}^2 = \mathsf{e}$.

Definition 3.7. $B_c := \mathsf{e}H_c\mathsf{e}$ is called *the spherical subalgebra* of H_c. The spherical subalgebra of $H_{t,c}$ is $B_{t,c} := B_c/(\hbar - t) = \mathsf{e}H_{t,c}\mathsf{e}$.

Note that

$$\mathsf{e}\,(\mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}))\,\mathsf{e} = \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^G, \quad \mathsf{e}\,(\mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*])\,\mathsf{e} = \mathbb{C}[\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*]^G.$$

Therefore, the restriction gives the embeddings: $\Theta_{1,c} : B_{1,c} \hookrightarrow \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^G$, and $\Theta_{0,c} : B_{0,c} \hookrightarrow \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{\mathrm{reg}}]^G$. In particular, we have

Proposition 3.8. *The spherical subalgebra $B_{0,c}$ is commutative and does not have zero divisors. Also $B_{0,c}$ is finitely generated.*

Proof. The first statement is clear from the above. The second statement follows from the fact that $\mathrm{gr}(B_{0,c}) = B_{0,0} = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^G$, which is finitely generated by Hilbert's theorem. $\square$

Corollary 3.9. $M_c = \mathrm{Spec} B_{0,c}$ *is an irreducible affine algebraic variety.*

Proof. Directly from the definition and the proposition. $\square$

We also obtain

Proposition 3.10. B_c *is a flat quantization (non-commutative deformation) of* $B_{0,c}$ *over* $\mathbb{C}[\hbar]$.

So $B_{0,c}$ carries a Poisson bracket $\{\cdot,\cdot\}$(thus M_c is a Poisson variety), and B_c is a quantization of the Poisson bracket, i.e. if $a, b \in B_c$ and a_0, b_0 are the corresponding elements in $B_{0,c}$, then

$$[a, b]/\hbar \equiv \{a_0, b_0\} \pmod{\hbar}.$$

Definition 3.11. The Poisson variety M_c is called *the Calogero-Moser space* of $G, \mathfrak{h}$ with parameter c.

3.4. **The localization lemma.** Let $H^{loc}_{t,c} = H_{t,c}[\delta^{-1}]$ be the localization of $H_{t,c}$ as a module over $\mathbb{C}[\mathfrak{h}]$ with respect to the discriminant δ (a polynomial vanishing to the first order on each reflection plane). Define also $B^{\mathrm{loc}}_{t,c} = \mathsf{e} H^{\mathrm{loc}}_{t,c}\mathsf{e}$.

Proposition 3.12. (i) *For* $t \neq 0$ *the map* $\Theta_{t,c}$ *induces an isomorphism of algebras* $H^{loc}_{t,c} \to \mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$, *which restricts to an isomorphism* $B^{\mathrm{loc}}_{t,c} \to \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^G$.

(ii) *The map* $\Theta_{0,c}$ *induces an isomorphism of algebras* $H^{\mathrm{loc}}_{0,c} \to \mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{\mathrm{reg}}]$, *which restricts to an isomorphism* $B^{\mathrm{loc}}_{0,c} \to \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{\mathrm{reg}}]^G$.

Proof. This follows immediately from the fact that the Dunkl operators have poles only on the reflection hyperplanes. $\square$

Since $\mathrm{gr}(B_{0,c}) = B_{0,0} = \mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}]^G$, we get the following geometric corollary.

Corollary 3.13. (i) *The family of Poisson varieties* M_c *is a flat deformation of the Poisson variety* $M_0 := (\mathfrak{h} \times \mathfrak{h}^*)/G$. *In particular,* M_c *is smooth outside of a subset of codimension* 2.

(ii) *We have a natural map* $\beta_c : M_c \to \mathfrak{h}/G$, *such that* $\beta_c^{-1}(\mathfrak{h}_{\mathrm{reg}}/G)$ *is isomorphic to* $(\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*)/G$. *The Poisson structure on* M_c *is obtained by extension of the symplectic Poisson structure on* $(\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*)/G$.

Example 3.14. Let $W = \mathbb{Z}_2$, $\mathfrak{h} = \mathbb{C}$. Then $B_{0,c} = \langle x^2, xp, p^2 - c^2/x^2\rangle$. Let $X := x^2, Z := xp$ and $Y := p^2 - c^2/x^2$. Then $Z^2 - XY = c^2$. So M_c is isomorphic to the quadric $Z^2 - XY = c^2$ in the 3-dimensional space and it is smooth for $c \neq 0$.

3.5. **Category $\mathcal{O}$ for rational Cherednik algebras.** From the PBW theorem, we see that $H_{1,c} = S\mathfrak{h}^* \otimes \mathbb{C}G \otimes S\mathfrak{h}$. It is similar to the structure of the universal enveloping algebra of a simple Lie algebra: $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$. Namely, the subalgebra $\mathbb{C}G$ plays the role of the Cartan subalgebra, and the subalgebras $S\mathfrak{h}^*$ and $S\mathfrak{h}$ play the role of the positive and negative nilpotent subalgebras. This similarity allows one to define and study the category $\mathcal{O}$ analogous to the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for simple Lie algebras.

Definition 3.15. The category $\mathcal{O}_c(G, \mathfrak{h})$ is the category of modules over $H_{1,c}(G, \mathfrak{h})$ which are finitely generated over $S\mathfrak{h}^*$ and locally finite under $S\mathfrak{h}$ (i.e., for $M \in \mathcal{O}_c(G, \mathfrak{h})$, $\forall v \in M$, $(S\mathfrak{h})v$ is finite dimensional).

If M is a locally finite $(S\mathfrak{h})^G$-module, then

$$M = \oplus_{\lambda \in \mathfrak{h}^*/G} M_\lambda,$$

where

$$M_\lambda = \{v \in M | \forall p \in (S\mathfrak{h})^G, \exists N \, s.t. \, (p - \lambda(p))^N v = 0\},$$

(notice that $\mathfrak{h}^*/G = \mathrm{Specm}(S\mathfrak{h})^G$).

Proposition 3.16. *M_λ are $H_{1,c}$-submodules.*

Proof. Note first that we have an isomorphism $\mu : H_{1,c}(G, \mathfrak{h}) \cong H_{1,c}(G, \mathfrak{h}^*)$, which is given by $x_a \mapsto y_a, y_b \mapsto -x_b, g \mapsto g$. Now let $x_1, \dots, x_r$ be a basis of $\mathfrak{h}^*$ and $y_1, \dots, y_r$ a basis of $\mathfrak{h}$. Suppose $P = P(x_1, \dots, x_r) \in (S\mathfrak{h}^*)^G$. Then we have

$$[y, P] = \frac{\partial}{\partial y} P \in S\mathfrak{h}^*, \text{ where } y \in \mathfrak{h},$$

(this follows from the fact that both sides act in the same way in the polynomial representation, which is faithful). So using the isomorphism μ, we conclude that if $Q \in (S\mathfrak{h})^G, Q = Q(y_1, \dots, y_r)$, then $[x, Q] = -\partial_x Q$ for $x \in \mathfrak{h}^*$.

Now, to prove the proposition, the only thing we need to check is that M_λ is invariant under $x \in \mathfrak{h}^*$. For any $v \in M_\lambda$, we have $(Q - \lambda(Q))^N v = 0$ for some N. Then

$$(Q - \lambda(Q))^{N+1} xv = (N + 1)\partial_x Q \cdot (Q - \lambda(Q))^N v = 0.$$

So $xv \in M_\lambda$.

□

Corollary 3.17. *We have the following decomposition:*

$$\mathcal{O}_c(G, \mathfrak{h}) = \bigoplus_{\lambda \in \mathfrak{h}^*/G} \mathcal{O}_c(G, \mathfrak{h})_\lambda,$$

where $\mathcal{O}_c(G, \mathfrak{h})_\lambda$ is the subcategory of modules where $(S\mathfrak{h})^G$ acts with generalized eigenvalue λ.

Proof. Directly from the definition and the proposition. □

Note that $\mathcal{O}_c(G, \mathfrak{h})_\lambda$ is an abelian category closed under taking subquotients and extensions.

3.6. **The grading element.** Let

$$\mathbf{h} = \sum_i x_i y_i + \frac{1}{2} \dim \mathfrak{h} - \sum_{s \in \mathcal{S}} \frac{2c_s}{1 - \lambda_s} s. \tag{3.2}$$

Proposition 3.18. *We have*

$$[\mathbf{h}, x] = x, \, x \in \mathfrak{h}^*, \quad [\mathbf{h}, y] = -y, \, y \in \mathfrak{h}.$$

Proof. Let us prove the first relation; the second one is proved similarly. We have

$$\begin{aligned}[\mathbf{h}, x] &= \sum_i x_i[y_i, x] - \sum_{s\in\mathcal{S}} \frac{2c_s}{1-\lambda_s} \cdot \frac{\lambda_s - 1}{2}(\alpha_s^\vee, x)\alpha_s \cdot s \\ &= \sum_i x_i(y_i, x) - \sum_i x_i \sum_{s\in\mathcal{S}} c_s(\alpha_s^\vee, x)(\alpha_s, y_i)s + \sum_{s\in\mathcal{S}} c_s(\alpha_s^\vee, x)\alpha_s \cdot s.\end{aligned}$$

The last two terms cancel since $\sum_i x_i(\alpha_s, y_i) = \alpha_s$, so we get $\sum_i x_i(y_i, x) = x$. ☐

Proposition 3.19. *Let $G = W$ be a real reflection group. Let*

$$\mathbf{h} = \sum_i x_i y_i + \frac{1}{2}\dim\mathfrak{h} - \sum_{s\in\mathcal{S}} c_s s, \quad \mathbf{E} = -\frac{1}{2}\sum_i x_i^2, \quad \mathbf{F} = \frac{1}{2}\sum_i y_i^2.$$

Then

(i) $\mathbf{h} = \sum_i (x_i y_i + y_i x_i)/2$;
(ii) $\mathbf{h}, \mathbf{E}, \mathbf{F}$ *form an* $\mathfrak{sl}_2$*-triple.*

Proof. A direct calculation. ☐

Theorem 3.20. *Let M be a module over $H_{1,c}(G, \mathfrak{h})$.*

(i) *If $\mathfrak{h}$ acts locally nilpotently on M, then $\mathbf{h}$ acts locally finitely on M.*
(ii) *If M is finitely generated over $S\mathfrak{h}^*$, then $M \in \mathcal{O}_c(G, \mathfrak{h})_0$ if and only if $\mathbf{h}$ acts locally finitely on M.*

Proof. (i) Assume that $S\mathfrak{h}$ acts locally nilpotently on M. Let $v \in M$. Then $S\mathfrak{h} \cdot v$ is a finite dimensional vector space and let $d = \dim S\mathfrak{h} \cdot v$. We prove that v is $\mathbf{h}$-finite by induction in dimension d. We can use $d = 0$ as base, so only need to do the induction step. The space $S\mathfrak{h} \cdot v$ must contain a nonzero vector u such that $y \cdot u = 0$, $\forall y \in \mathfrak{h}$. Let $U \subset M$ be the subspace of vectors with this property. $\mathbf{h}$ acts on U by an element of $\mathbb{C}G$, hence locally finitely. So it is sufficient to show that the image of v in $M/\langle U\rangle$ is $\mathbf{h}$-finite (where $\langle U\rangle$ is the submodule generated by U). But this is true by the induction assumption, as $u = 0$ in $M/\langle U\rangle$.

(ii) We need to show that if $\mathbf{h}$ acts locally finitely on M, then $\mathfrak{h}$ acts locally nilpotently on M. Assume $\mathbf{h}$ acts locally finitely on M. Then $M = \oplus_{\beta\in B} M[\beta]$, where $B \subset \mathbb{C}$. Since M is finitely generated over $S\mathfrak{h}^*$, B is a finite union of sets of the form $z + \mathbb{Z}_{\geq 0}$, $z \in \mathbb{C}$. So $S\mathfrak{h}$ must act locally nilpotently on M. ☐

We can obtain the following corollary easily.

Corollary 3.21. *Any finite dimensional $H_{1,c}(G, \mathfrak{h})$-module is in $\mathcal{O}_c(G, \mathfrak{h})_0$.*

We see that any module $M \in \mathcal{O}_c(G, \mathfrak{h})_0$ has a grading by generalized eigenvalues of $\mathbf{h}$: $M = \oplus_\beta M[\beta]$.

3.7. **Standard modules.** Let τ be a finite dimensional representation of G. The standard module over $H_{1,c}(G, \mathfrak{h})$ corresponding to τ (also called the Verma module) is

$$M_c(G, \mathfrak{h}, \tau) = M_c(\tau) = H_{1,c}(G, \mathfrak{h}) \otimes_{\mathbb{C}G\ltimes S\mathfrak{h}} \tau \ \in \mathcal{O}_c(G, \mathfrak{h})_0,$$

where $S\mathfrak{h}$ acts on τ by zero.

So from the PBW theorem, we have that as vector spaces, $M_c(\tau) \cong \tau \otimes S\mathfrak{h}^*$.

Remark 3.22. More generally, $\forall \lambda \in \mathfrak{h}^*$, let $G_\lambda = \mathrm{Stab}(\lambda)$, and τ be a finite dimensional representation of G_λ. Then we can define $M_{c,\lambda}(G, \mathfrak{h}, \tau) = H_{1,c}(G, \mathfrak{h}) \otimes_{\mathbb{C}G_\lambda \ltimes S\mathfrak{h}} \tau$, where $S\mathfrak{h}$ acts on τ by λ. These modules are called *the Whittaker modules.*

Let τ be irreducible, and let $h_c(\tau)$ be the number given by the formula

$$h_c(\tau) = \frac{\dim \mathfrak{h}}{2} - \sum_{s \in \mathcal{S}} \frac{2c_s}{1 - \lambda_s} s|_\tau.$$

Then we see that $\mathbf{h}$ acts on $\tau \otimes S^m \mathfrak{h}^*$ by the scalar $h_c(\tau) + m$.

Definition 3.23. A vector v in an $H_{1,c}$-module M is singular if $y_i v = 0$ for all i.

Proposition 3.24. *Let U be an $H_{1,c}(G, \mathfrak{h})$-module. Let $\tau \subset U$ be a G-submodule consisting of singular vectors. Then there is a unique homomorphism $\phi : M_c(\tau) \to U$ of $\mathbb{C}[\mathfrak{h}]$-modules such that $\phi|_\tau$ is the identity, and it is an $H_{1,c}$-homomorphism.*

Proof. The first statement follows from the fact that $M_c(\tau)$ is a free module over $\mathbb{C}[\mathfrak{h}]$ generated by τ. Also, it follows from the Frobenius reciprocity that there must exist a map ϕ which is an $H_{1,c}$-homomorphism. This implies the proposition. $\square$

3.8. **Finite length.**

Proposition 3.25. *$\exists K \in \mathbb{R}$ such that for any $M \subset N$ in $\mathcal{O}_c(G, \mathfrak{h})_0$, if $M[\beta] = N[\beta]$ for* $\mathrm{Re}\,(\beta) \leq K$, *then $M = N$.*

Proof. Let $K = \max_\tau \mathrm{Re}\, h_c(\tau)$. Then if $M \neq N$, M/N begins in degree β_0 with $\mathrm{Re}\,\beta_0 > K$, which is impossible since by Proposition 3.24, β_0 must equal $h_c(\tau)$ for some τ. $\square$

Corollary 3.26. *Any $M \in \mathcal{O}_c(G, \mathfrak{h})_0$ has finite length.*

Proof. Directly from the proposition. $\square$

3.9. **Characters.** For $M \in \mathcal{O}_c(G, \mathfrak{h})_0$, define the character of M as the following formal series in t:

$$\mathrm{ch}_M(g, t) = \sum_\beta t^\beta \mathrm{Tr}_{M[\beta]}(g) = \mathrm{Tr}_M(g t^{\mathbf{h}}), \quad g \in G.$$

Proposition 3.27. *We have*

$$\mathrm{ch}_{M_c(\tau)}(g, t) = \frac{\chi_\tau(g) t^{h_c(\tau)}}{\det_{\mathfrak{h}^*}(1 - tg)}.$$

Proof. We begin with the following lemma.

Lemma 3.28 (MacMahon's Master theorem)**.** *Let V be a finite dimensional space, $A : V \to V$ a linear operator. Then*

$$\sum_{n \geq 0} t^n \mathrm{Tr}\,(S^n A) = \frac{1}{\det(1 - tA)}.$$

Proof of the lemma. If A is diagonalizable, this is obvious. The general statement follows by continuity. $\square$

The lemma implies that $\mathrm{Tr}_{S\mathfrak{h}^*}(g t^D) = \dfrac{1}{\det(1 - gt)}$ where D is the degree operator. This implies the required statement. $\square$

3.10. **Irreducible modules.** Let τ be an irreducible representation of G.

Proposition 3.29. *$M_c(\tau)$ has a maximal proper submodule $J_c(\tau)$.*

Proof. The proof is standard. $J_c(\tau)$ is the sum of all proper submodules of $M_c(\tau)$, and it is not equal to $M_c(\tau)$ because any proper submodule has a grading by generalized eigenspaces of $\mathbf{h}$, with eigenvalues β such that $\beta - h_c(\tau) > 0$. $\square$

We define $L_c(\tau) = M_c(\tau)/J_c(\tau)$, which is an irreducible module.

Proposition 3.30. *Any irreducible object of $\mathcal{O}_c(G, \mathfrak{h})_0$ has the form $L_c(\tau)$ for an unique τ.*

Proof. Let $L \in \mathcal{O}_c(G, \mathfrak{h})_0$ be irreducible, with lowest eigenspace of $\mathbf{h}$ containing an irreducible G-module τ. Then by Proposition 3.24, we have a nonzero homomorphism $M_c(\tau) \to L$, which is surjective, since L is irreducible. Then we must have $L = L_c(\tau)$. $\square$

Remark 3.31. Let χ be a character of G. Then we have an isomorphism $H_{1,c}(G, \mathfrak{h}) \to H_{1,c\chi}(G, \mathfrak{h})$, mapping $g \in G$ to $\chi^{-1}(g)g$. This automorphism maps $L_c(\tau)$ to $L_{c\chi}(\chi^{-1} \otimes \tau)$ isomorphically.

3.11. **The contragredient module.** Set $\bar{c}(s) = c(s^{-1})$. We have a natural isomorphism $\gamma : H_{1,\bar{c}}(G, \mathfrak{h}^*)^{\mathrm{op}} \to H_{1,c}(G, \mathfrak{h})$, acting trivially on $\mathfrak{h}$ and $\mathfrak{h}^*$, and sending $g \in G$ to g^{-1}.

Thus if M is an $H_{1,c}(G, \mathfrak{h})$-module, then the full dual space M^* is an $H_{1,\bar{c}}(M, \mathfrak{h}^*)$-module. If $M \in \mathcal{O}_c(G, \mathfrak{h})_0$, then we can define $M^\dagger$, which is the $\mathbf{h}$-finite part of M^*.

Proposition 3.32. *$M^\dagger$ belongs to $\mathcal{O}_{\bar{c}}(G, \mathfrak{h}^*)_0$.*

Proof. Clearly, if L is irreducible, then so is $L^\dagger$. Then $L^\dagger$ is generated by its lowest $\mathbf{h}$-eigenspace over $H_{1,\bar{c}}(G, \mathfrak{h}^*)$, hence over $S\mathfrak{h}^*$. Thus, $L^\dagger \in \mathcal{O}_{\bar{c}}(G, \mathfrak{h}^*)_0$. Now, let $M \in \mathcal{O}_c(G, \mathfrak{h})_0$ be any object. Since M has finite length, so does $M^\dagger$. Moreover, $M^\dagger$ has a finite filtration with successive quotients of the form $L^\dagger$, where $L \in \mathcal{O}_c(G, \mathfrak{h})_0$ is irreducible. This implies the required statement, since $\mathcal{O}_c(G, \mathfrak{h})_0$ is closed under taking extensions. $\square$

Clearly, $M^{\dagger\dagger} = M$. Thus, $M \mapsto M^\dagger$ is an equivalence of categories $\mathcal{O}_c(G, \mathfrak{h}) \to \mathcal{O}_{\bar{c}}(G, \mathfrak{h}^*)^{\mathrm{op}}$.

3.12. **The contravariant form.** Let τ be an irreducible representation of G. By Proposition 3.24, we have a unique homomorphism $\phi : M_c(G, \mathfrak{h}, \tau) \to M_{\bar{c}}(G, \mathfrak{h}^*, \tau^*)^\dagger$ which is the identity in the lowest $\mathbf{h}$-eigenspace. Thus, we have a pairing

$$\beta_c : M_c(G, \mathfrak{h}, \tau) \times M_{\bar{c}}(G, \mathfrak{h}^*, \tau^*) \to \mathbb{C},$$

which is called *the contravariant form*.

Remark 3.33. If $G = W$ is a real reflection group, then $\mathfrak{h} \cong \mathfrak{h}^*$, $c = \bar{c}$, and $\tau \cong \tau^*$ via a symmetric form. So for real reflection groups, β_c is a symmetric form on $M_c(\tau)$.

Proposition 3.34. *The maximal proper submodule $J_c(\tau)$ is the kernel of ϕ (or, equivalently, of the contravariant form β_c).*

Proof. Let K be the kernel of the contravariant form. It suffices to show that $M_c(\tau)/K$ is irreducible. We have a diagram:

$$\begin{tikzcd} M_c(G, \mathfrak{h}, \tau) \arrow[r, "\phi"] \arrow[d] \arrow[dr, "\xi"] & M_c(G, \mathfrak{h}^*, \tau^*)^\dagger \\ L_c(G, \mathfrak{h}, \tau) \arrow[r, "\sim", "\eta"'] & L_c(G, \mathfrak{h}^*, \tau^*)^\dagger \arrow[u, hook] \end{tikzcd}$$

Indeed, a nonzero map ξ exists by Proposition 3.24, and it factors through $L_c(G,\mathfrak{h},\tau)$, with η being an isomorphism, since $L_c(G,\mathfrak{h}^*,\tau^*)^\dagger$ is irreducible. Now, by Proposition 3.24 (uniqueness of ϕ), the diagram must commute up to scaling, which implies the statement. $\square$

Proposition 3.35. *Assume that $h_c(\tau)-h_c(\tau')$ never equals a positive integer for any $\tau,\tau'\in$ Irrep$G$. Then $\mathcal{O}_c(G,\mathfrak{h})_0$ is semisimple, with simple objects $M_c(\tau)$.*

Proof. It is clear that in this situation, all $M_c(\tau)$ are simple. Also consider $\mathrm{Ext}^1(M_c(\tau),M_c(\tau'))$. If $h_c(\tau)-h_c(\tau')\notin\mathbb{Z}$, it is clearly 0. Otherwise, $h_c(\tau)=h_c(\tau')$, and again $\mathrm{Ext}^1(M_c(\tau),M_c(\tau'))=0$, since for any extension

$$0\to M_c(\tau')\to N\to M_c(\tau)\to 0,$$

by Proposition 3.24 we have a splitting $M_c(\tau)\to N$. $\square$

Remark 3.36. In fact, our argument shows that if $\mathrm{Ext}^1(M_c(\tau),M_c(\tau'))\neq 0$, then $h_c(\tau)-h_c(\tau')\in\mathbb{N}$.

3.13. **The matrix of multiplicities.** For $\tau,\sigma\in \mathrm{I}rrepG$, write $\tau<\sigma$ if

$$\mathrm{Re}\,h_c(\sigma)-\mathrm{Re}\,h_c(\tau)\in\mathbb{N}.$$

Proposition 3.37. *There exists a matrix of integers $N=(n_{\sigma,\tau})$, with $n_{\sigma,\tau}\geq 0$, such that $n_{\tau,\tau}=1$, $n_{\sigma,\tau}=0$ unless $\sigma<\tau$, and*

$$M_c(\sigma)=\sum n_{\sigma,\tau}L_c(\tau)\ \in \mathrm{K}_0(\mathcal{O}_c(G,\mathfrak{h})_0).$$

Proof. This follows from the Jordan-Hölder theorem and the fact that objects in $\mathcal{O}_c(G,\mathfrak{h})_0$ have finite length. $\square$

Corollary 3.38. *Let $N^{-1}=(\bar{n}_{\tau,\sigma})$. Then*

$$L_c(\tau)=\sum\bar{n}_{\tau,\sigma}M_c(\sigma).$$

Corollary 3.39. *We have*

$$\mathrm{ch}_{\,L_c(\tau)}(g,t)=\frac{\sum\bar{n}_{\tau,\sigma}\chi_\sigma(g)t^{h_c(\tau)}}{\det_{\mathfrak{h}^*}(1-tg)}.$$

Both of the corollaries can be obtained from the above proposition easily.

One of the main problems in the representation theory of rational Cherednik algebras is the following problem.

Problem: Compute the multiplicities $n_{\sigma,\tau}$ or, equivalently, $\mathrm{ch}_{\,L_c(\tau)}$ for all τ.

In general, this problem is open.

3.14. **Example: the rank 1 case.** Let $G=\mathbb{Z}/m\mathbb{Z}$ and λ be an m-th primitive root of 1. Then the algebra $H_{1,c}(G,\mathfrak{h})$ is generated by x,y,s with relations

$$[y,x]=1-2\sum_{j=1}^{m-1}c_js^j,\quad sxs^{-1}=\lambda x,\quad sys^{-1}=\lambda^{-1}y.$$

Consider the one-dimensional space $\mathbb{C}$ and let y act by 0 and $g\in G$ act by 1. We have $M_c(\mathbb{C})=\mathbb{C}[x]$. The contravariant form $\beta_{c,\mathbb{C}}$ on $M_c(\mathbb{C})$ is defined by

$$\beta_{c,\mathbb{C}}(x^n,x^n)=a_n;\quad \beta_{c,\mathbb{C}}(x^n,x^{n'})=0, n\neq n'.$$

Recall that $\beta_{c,\mathbb{C}}$ satisfies $\beta_{c,\mathbb{C}}(x^n, x^n) = \beta_{c,\mathbb{C}}(x^{n-1}, yx^n)$, which gives

$$a_n = a_{n-1}(n - b_n),$$

where b_n are new parameters:

$$b_n := 2\sum_{j=1}^{m-1} \frac{1-\lambda^{jn}}{1-\lambda^j} c_j \quad (b_0 = 0, b_{n+m} = b_n).$$

Thus we obtain the following proposition.

Proposition 3.40. (i) *$M_c(\mathbb{C})$ is irreducible if only if $n - b_n \neq 0$ for any $n \geq 1$.*

(ii) *Assume that r is the smallest positive integer such that $r = b_r$. Then $L_c(\mathbb{C})$ has dimension r (which can be any number not divisible by m) with basis $1, x, \ldots, x^{r-1}$.*

Remark 3.41. According to Remark 3.31, this proposition in fact describes all the irreducible lowest weight modules.

Example 3.42. Consider the case $m = 2$. The $M_c(\mathbb{C})$ is irreducible unless $c \in 1/2 + \mathbb{Z}_{\geq 0}$. If $c = (2n+1)/2 \in 1/2 + \mathbb{Z}$, $n \geq 0$, then $L_c(\mathbb{C})$ has dimension $2n + 1$. A similar answer is obtained for lowest weight $\mathbb{C}_-$, replacing c by $-c$.

3.15. **The Frobenius property.** Let A be a $\mathbb{Z}_+$-graded commutative algebra. The algebra A is called Frobenius if the top degree $A[d]$ of A is 1-dimensional, and the multiplication map $A[m] \times A[d-m] \to A[d]$ is a nondegenerate pairing for any $0 \leq m \leq d$. In particular, the Hilbert polynomial of a Frobenius algebra A is palindromic.

Now, let us go back to considering modules over the rational Cherednik algebra $H_{1,c}$. Any submodule J of the polynomial representation $M_c(\mathbb{C}) = M_c = \mathbb{C}[\mathfrak{h}]$ is an ideal in $\mathbb{C}[\mathfrak{h}]$, so the quotient $A = M_c/J$ is a $\mathbb{Z}_+$-graded commutative algebra.

Now suppose that G preserves an inner product in $\mathfrak{h}$, i.e., $G \subseteq \mathrm{O}(\mathfrak{h})$.

Theorem 3.43. *If $A = M_c(\mathbb{C})/J$ is finite dimensional, then A is irreducible ($A = L_c(\mathbb{C})$) $\Longleftrightarrow$ A is a Frobenius algebra.*

Proof. 1) Suppose A is an irreducible $H_{1,c}$-module, i.e., $A = L_c(\mathbb{C})$. By Proposition 3.19, A is naturally a finite dimensional $\mathfrak{sl}_2$-module (in particular, it integrates to the group $\mathrm{SL}_2(\mathbb{C})$). Hence, by the representation theory of $\mathfrak{sl}_2$, the top degree of A is 1-dimensional. Let $\phi \in A^*$ denote a nonzero linear function on the top component. Let β_c be the contravariant form on $M_c(\mathbb{C})$. Consider the form

$$(v_1, v_2) \mapsto E(v_1, v_2) := \beta_c(v_1, gv_2), \text{ where } g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

Then $E(xv_1, v_2) = E(v_1, xv_2)$. So for any $p, q \in M_c(\mathbb{C}) = \mathbb{C}[\mathfrak{h}]$, $E(p, q) = \phi(p(x)q(x))$ (for a suitable normalization of ϕ).

Since E is a nondegenerate form, A is a Frobenius algebra.

2) Suppose A is Frobenius. Then the highest component is 1-dimensional, and $E : A \otimes A \to \mathbb{C}$, $E(a, b) = \phi(ab)$ is nondegenerate. We have $E(xa, b) = E(a, xb)$. So set $\beta(a, b) = E(a, g^{-1}b)$. Then β satisfies $\beta(a, x_i b) = \beta(y_i a, b)$. Thus, for all $p, q \in \mathbb{C}[\mathfrak{h}]$, $\beta(p(x), q(x)) = \beta(q(y)p(x), 1)$. So $\beta = \beta_c$ up to scaling. Thus, β_c is nondegenerate and A is irreducible. $\square$

Remark 3.44. If $G \not\subseteq \mathrm{O}(\mathfrak{h})$, this theorem is false, in general.

Now consider the Frobenius property of $L_c(\mathbb{C})$ for any $G \subset \mathrm{GL}(\mathfrak{h})$.

Theorem 3.45. *Let $U \subset M_c(\mathbb{C}) = \mathbb{C}[\mathfrak{h}]$ be a G-subrepresentation of dimension $l = \dim \mathfrak{h}$, sitting in degree r, which consists of singular vectors. Let $J = \langle U \rangle$. Assume that $A = M_c/J$ is finite dimensional. Then*

(i) *A is Frobenius.*

(ii) *A admits a BGG type resolution:*

$$A \leftarrow M_c(\mathbb{C}) \leftarrow M_c(U) \leftarrow M_c(\wedge^2 U) \leftarrow \cdots \leftarrow M_c(\wedge^l U) \leftarrow 0.$$

(iii) *The character of A is given by the formula*

$$\chi_A(g,t) = t^{\frac{l}{2} - \sum_{s\in S} \frac{2c_s}{1-\lambda_s}} \frac{\det_U(1 - gt^r)}{\det_{\mathfrak{h}^*}(1-gt)}.$$

In particular, $\dim A = r^l$.

(iv) *If G preserves an inner product, then A is irreducible.*

Proof. (i) Since $\operatorname{Spec} A$ is a complete intersection, A is Frobenius.

(ii) We will use the following theorem:

Theorem 3.46 (Serre). *Let $f_1, \dots, f_n \in \mathbb{C}[t_1, \dots, t_n]$ be homogeneous polynomials, and assume that $\mathbb{C}[t_1, \dots, t_n]$ is a finitely generated module over $\mathbb{C}[f_1, \dots, f_n]$. Then this is a free module.*

Consider $SU \subset S\mathfrak{h}^*$. Then $S\mathfrak{h}^*$ is a finitely generated SU-module (as $S\mathfrak{h}^*/\langle U \rangle$ is finite dimensional). By Serre's theorem, we know that $S\mathfrak{h}^*$ is a free SU-module. The rank of this module is r^l. Consider the Koszul complex attached to this module. Since the module is free, the Koszul complex is exact (i.e., it is a resolution of the zero fiber). At the level of SU-modules, it looks exactly like we want in (3.45).

So we only need to show that the maps of the resolution are morphisms over $H_{1,c}$. This is shown by induction. Namely, let $\delta_j : M_c(\wedge^j U) \to M_c(\wedge^{j-1} U)$ be the corresponding differentials (so that $\delta_0 : M_c(\mathbb{C}) \to A$ is the projection). Then δ_0 is an $H_{1,c}$-morphism, which is the base of induction. If δ_j is an $H_{1,c}$-morphism, then the kernel of δ_j is a submodule $K_j \subset M_c(\wedge^j U)$. Its lowest degree part is $\wedge^{j+1} U$ sitting in degree $(j+1)r$ and consisting of singular vectors. Now, δ_{j+1} is a morphism over $S\mathfrak{h}^*$ which maps $\wedge^{j+1} U$ identically to itself. By Proposition 3.24, there is only one such morphism, and it must be an $H_{1,c}$-morphism. This completes the induction step.

(iii) follows from (ii) by the Euler-Poincaré formula.

(iv) follows from Theorem 3.43.

□

3.16. **Representations of $H_{1,c}$ of type A.** Let us now apply the above results to the case of type A. We will follow the paper [CE].

Let $G = \mathfrak{S}_n$, and $\mathfrak{h}$ be its reflection representation. In this case the function c reduces to one number. We will denote the rational Cherednik algebra $H_{1,c}(\mathfrak{S}_n)$ by $H_c(n)$. It is generated by $x_1, \dots, x_n$, $y_1, \dots, y_n$ and $\mathbb{C}\mathfrak{S}_n$ with the following relations:

$$\sum y_i = 0, \quad \sum x_i = 0, \quad [y_i, x_j] = -\frac{1}{n} + c s_{ij}, i \neq j,$$

$$[y_i, x_i] = \frac{n-1}{n} - c\sum_{j\neq i} s_{ij}.$$

The polynomial representation $M_c(\mathbb{C})$ of this algebra is the space of $\mathbb{C}[x_1, \ldots, x_n]^T$ of polynomials of $x_1, \ldots, x_n$, which are invariant under simultaneous translation $T : x_i \mapsto x_i + a$. In other words, it is the space of regular functions on $\mathfrak{h} = \mathbb{C}^n/\Delta$, where Δ is the diagonal.

Proposition 3.47 (C. Dunkl)**.** *Let r be a positive integer not divisible by n, and $c = r/n$. Then $M_c(\mathbb{C})$ contains a copy of the reflection representation $\mathfrak{h}$ of $\mathfrak{S}_n$, which consists of singular vectors (i.e. those killed by $y \in \mathfrak{h}$). This copy sits in degree r and is spanned by the functions*

$$f_i(x_1, \ldots, x_n) = \mathrm{Res}_\infty[(z - x_1)\cdots(z - x_n)]^{\frac{r}{n}} \frac{\mathrm{d}z}{z - x_i}.$$

(the symbol Res_∞ *denotes the residue at infinity).*

Remark 3.48. The space spanned by f_i is $(n-1)$-dimensional, since $\sum_i f_i = 0$ (this sum is the residue of an exact differential).

Proof. This proposition can be proved by a straightforward computation. The functions f_i are a special case of Jack polynomials. ☐

Let I_c be the submodule of $M_c(\mathbb{C})$ generated by f_i. Consider the $H_c(n)$-module $V_c = M_c(\mathbb{C})/I_c$, and regard it as a $\mathbb{C}[\mathfrak{h}]$-module. We have the following results.

Theorem 3.49. *Let $d = (r, n)$ denote the greatest common divisor of r and n. Then the (set-theoretical) support of V_c is the union of $\mathfrak{S}_n$-translates of the subspaces of $\mathbb{C}^n/\Delta$, defined by the equations*

$$x_1 = x_2 = \cdots = x_{\frac{n}{d}}; \quad x_{\frac{n}{d}+1} = \cdots = x_{2\frac{n}{d}}; \quad \ldots \quad x_{(d-1)\frac{n}{d}+1} = \cdots = x_n.$$

In particular, the Gelfand-Kirillov dimension of V_c is $d - 1$.

Corollary 3.50 ([BEG])**.** *If $d = 1$ then the module V_c is finite dimensional, irreducible, admits a BGG type resolution, and its character is*

$$\chi_{V_c}(g, t) = t^{(1-r)(n-1)/2} \frac{\det|_{\mathfrak{h}}(1 - gt^r)}{\det|_{\mathfrak{h}}(1 - gt)}.$$

Proof. For $d = 1$ Theorem 3.49 says that the support of $M_c(\mathbb{C})/I_c$ is $\{0\}$. This implies that $M_c(\mathbb{C})/I_c$ is finite dimensional. The rest follows from Theorem 3.45. ☐

Proof of Theorem 3.49. The support of V_c is the zero set of I_c, i.e. the common zero set of f_i. Fix $x_1, \ldots, x_n \in \mathbb{C}$. Then $f_i(x_1, \ldots, x_n) = 0$ for all i iff $\sum_{i=1}^n \lambda_i f_i = 0$ for all λ_i, i.e.

$$\mathrm{Res}_\infty \left(\prod_{j=1}^n (z - x_j)^{\frac{r}{n}} \sum_{i=1}^n \frac{\lambda_i}{z - x_i} \right) \mathrm{d}z = 0.$$

Assume that $x_1, \ldots x_n$ take distinct values $y_1, \ldots, y_p$ with positive multiplicities $m_1, \ldots, m_p$. The previous equation implies that the point $(x_1, \ldots, x_n)$ is in the zero set iff

$$\mathrm{R}es_\infty \prod_{j=1}^p (z - y_j)^{m_j \frac{r}{n} - 1} \left(\sum_{i=1}^p \nu_i (z - y_1) \cdots \widehat{(z - y_i)} \cdots (z - y_p) \right) \mathrm{d}z = 0 \quad \forall \nu_i.$$

Since ν_i are arbitrary, this is equivalent to the condition

$$\mathrm{R}es_\infty \prod_{j=1}^{p}(z-y_j)^{m_j\frac{r}{n}-1}z^i\mathrm{d}z = 0, \quad i=0,\dots,p-1.$$

We will now need the following lemma.

Lemma 3.51. *Let* $a(z)=\prod_{j=1}^{p}(z-y_j)^{\mu_j}$*, where* $\mu_j\in\mathbb{C}$*,* $\sum_j\mu_j\in\mathbb{Z}$ *and* $\sum_j\mu_j>-p$*. Suppose*

$$\mathrm{R}es_\infty a(z)z^i\mathrm{d}z=0, \quad i=0,1,\dots,p-2.$$

Then $a(z)$ *is a polynomial.*

Proof. Let $g(z)$ be a polynomial. Then

$$0=\mathrm{R}es_\infty\mathrm{d}(g(z)\cdot a(z))=\mathrm{Res}_\infty(g'(z)a(z)+a'(z)g(z))\mathrm{d}z,$$

and hence

$$\mathrm{R}es_\infty\left(g'(z)+\sum_i\frac{\mu_j}{z-y_j}g(z)\right)a(z)\mathrm{d}z=0.$$

Let $g(z)=z^l\prod_j(z-y_j)$. Then $g'(z)+\sum_j\frac{\mu_j}{z-y_j}g(z)$ is a polynomial of degree $l+p-1$ with highest coefficient $l+p+\sum\mu_j\neq 0$ (as $\sum\mu_j>-p$). This means that for every $l\geq 0$, $\mathrm{R}es_\infty z^{l+p-1}a(z)\mathrm{d}z$ is a linear combination of residues of $z^qa(z)\mathrm{d}z$ with $q<l+p-1$. By the assumption of the lemma, this implies by induction in l that all such residues are 0 and hence a is a polynomial. $\square$

In our case $\sum(m_jr/n-1)=r-p$ (since $\sum m_j=n$) and the conditions of the lemma are satisfied. Hence $(x_1,\dots,x_n)$ is in the zero set of I_c iff $\prod_{j=1}^{p}(z-y_j)^{m_j\frac{r}{n}-1}$ is a polynomial. This is equivalent to saying that all m_j are divisible by n/d.

We have proved that $(x_1,\dots,x_n)$ is in the zero set of I_c if and only if $(z-x_1)\cdots(z-x_n)$ is the (n/d)-th power of a polynomial of degree d. This implies the theorem. $\square$

Remark 3.52. For $c>0$, the above representations are the only irreducible finite dimensional representations of $H_{1,c}(\mathfrak{S}_n)$. Namely, it is proved in [BEG] that the only finite dimensional representations of $H_{1,c}(\mathfrak{S}_n)$ are multiples of $L_c(\mathbb{C})$ for $c=r/n$, and of $L_c(\mathbb{C}_-)$ (where $\mathbb{C}_-$ is the sign representation) for $c=-r/n$, where r is a positive integer relatively prime to n.

3.17. **Notes.** The discussion of the definition of rational Cherednik algebras and their basic properties follows Section 7 of [E4]. The discussion of the category $\mathcal{O}$ for rational Cherednik algebras follows Section 11 of [E4]. The material in Sections 3.14-3.16 is borrowed from [CE].

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