THE KODAIRA DIMENSION OF THE MODULI SPACE OF CURVES

1. Preliminaries

A great reference for background about linear systems, big and ample line bundles and Kodaira dimensions is [L]. Here we will only develop a few basics that will be necessary for our discussion of the Kodaira dimension of the moduli space of curves.

Let L be a line bundle on a normal, irreducible, projective variety X. The semi-group $N(X, L)$ of L is defined to be the non-negative powers of L that have a non-zero section:

$$N(X, L) := \{ \ m \geq 0 \ : \ H^0(X, L^{\otimes m}) > 0 \ \}.$$

Given $m \in N(X, L)$ we can consider the rational map ϕ_m associated to $L^{\otimes m}$.

Definition 1.1. Let L be a line bundle on a normal, irreducible, projective variety. Then the Iitaka dimension of L is defined to be the maximum dimension of the image of ϕ_m for $m \in N(X, L)$ provided $N(X, L) \neq 0$. If $N(X, L) = 0$, then the Iitaka dimension of L is defined to be $-\infty$. When X is smooth, the Kodaira dimension of X is defined to be the Iitaka dimension of its canonical bundle K_X. If X is singular, the Kodaira dimension of X is defined to be the Kodaira dimension of any desingularization of X.

Remark 1.2. Note that by definition the Iitaka dimension of a line bundle L on X is an integer between 0 and $\dim(X)$ or it is $-\infty$.

Definition 1.3. A line bundle L on a normal, projective variety is called big if its Iitaka dimension is equal to the dimension of X. A smooth, projective variety is called of general type if its canonical bundle is big. A singular variety is called of general type if a desingularization is of general type.

Remark 1.4. Of course, the same definitions can be made for Cartier (or even $\mathbb{Q}$-Cartier) divisors instead of line bundles. Below we will use the language of Cartier divisors and line bundles interchangably.

An alternative definition of big line bundles in terms of cohomology is given by the following well-known lemma.

Lemma 1.5. *A line bundle L on a normal, projective variety X of dimension n is big if and only if there exists a positive constant C such that*

$$h^0(X, L^{\otimes m}) \geq Cm^n$$

for all sufficiently large $m \in N(X, L)$.

Kodaira's Lemma allows us to obtain other useful characterizations of big line bundles.

Lemma 1.6 (Kodaira's Lemma)**.** *Let D be a big Cartier divisor and E be an arbitrary effective Cartier divisor on a normal, projective variety X. Then*

$$H^0(X, \mathcal{O}_X(mD - E)) \neq 0$$

for all sufficiently large $m \in N(X, D)$.

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_X(mD - E) \to \mathcal{O}_X(mD) \to \mathcal{O}_E(mD) \to 0.$$

Since D is big by assumption, the dimension of global sections of $\mathcal{O}_X(mD)$ grows like $m^{\dim(X)}$. On the other hand, $\dim(E) < \dim(X)$, hence the dimension of global sections of $\mathcal{O}_E(mD)$ grows at most like $m^{\dim(X)-1}$. It follows that

$$h^0(X, \mathcal{O}_X(mD)) > h^0(E, \mathcal{O}_E(mD)$$

for large enough $m \in N(X, D)$. The lemma follows by the long exact sequence of cohomology associated to the exact sequence of sheaves. $\square$

A corollary of Kodaira's Lemma is the characterization of big divisors as those divisors that are numerically equivalent to the sum of an ample and an effective divisor. We will use this characterization in determining the Kodaira dimension of the moduli space of curves.

Proposition 1.7. *Let D be a divisor on a normal, irreducible projective variety X. Then the following are equivalent:*

(1) *D is big.*

(2) *For any ample divisor A, there exists an integer $m > 0$ and an effective divisor E such that mD is linearly equivalent to $A + E$.*

(3) *There exists an ample divisor A, an integer $m > 0$ and an effective divisor E such that mD is linearly equivalent to $A + E$.*

(4) *There exists an ample divisor A, an integer $m > 0$ and an effective divisor E such that mD is numerically equivalent to $A + E$.*

Proof. To prove that (1) implies (2) given any ample divisor A, take a large enough positive number r such that both rA and $(r+1)A$ are effective. By Kodaira's Lemma there is a positive integer m such that $mD - (r+1)A$ is effective, say linearly equivalent to an effective divisor E. We thus get that mD is linearly equivalent to $A + (rA + E)$ proving (2). Clearly (2) implies (3) and (3) implies (4). To see that (4) implies (1), since mD is numerically equivalent to $A+E$, $mD-E$ is numerically equivalent to an ample divisor. Since ampleness is numerical, $mD - E$ is ample. Since ample divisors are big and

$$h^0(X, mD) \geq h^0(X, mD - E),$$

D is big. $\square$

2. The canonical bundle of the moduli space of curves

We can calculate the canonical class of the moduli space of curves using the Grothendieck - Riemann - Roch formula.

Theorem 2.1. *The canonical class of the coarse moduli scheme $\overline{M}_g$ is given by*

$$K_{\overline{M}_g} = 13\lambda - 2\delta - \delta_1.$$

Proof. The cotangent bundle of $\overline{\mathrm{M}}_g$ at a smooth, automorphism-free curve is given by the space of quadratic differentials. More generally, over the automorphism-free locus the canonical bundle will be the first chern class of

$$\pi_*(\Omega_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g} \otimes \omega_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g}).$$

We can easily calculate this class in the Picard group of the moduli functor:

$$\pi_*\left((1 + c_1(\Omega \otimes \omega) + \frac{c_1^2(\Omega \otimes \omega)}{2} - c_2(\Omega \otimes \omega))(1 - \frac{c_1(\Omega)}{2} + \frac{c_1^2(\Omega) + c_2(\Omega)}{12})\right)$$

Expanding (and using the relations we proved in the last unit) we see that this expression equals

$$\pi_*\left(2c_1^2(\omega) - [Sing] - c_1^2(\omega) + \frac{c_1^2(\omega) + [Sing]}{12}\right) = 13\lambda - 2\delta.$$

We need to adjust this formula to take into account that every element of the locus of curves with an elliptic tail have an automorphism given by the hyperelliptic involution on the elliptic tail. The effect of this can be calculated in local coordinates to see that it introduces a simple zero along that locus. □

Remark 2.2. One word of caution is in order. Recall that δ_1 does not descend to the coarse moduli scheme because every curve in the boundary locus has an automorphism of order 2. However, δ_1^2 descends to the coarse moduli scheme. Accordingly we defined the class δ_1 as half of the class of the boundary locus Δ_1. In terms of the class of the loci of reducible curves the canonical class is

$$13\lambda - 2[\Delta] + \frac{1}{2}[\Delta_1].$$

3. Ample divisors on the moduli space of curves

In order to show that the moduli space is of general type we need to show that the canonical bundle is big (on a desingularization). In view of the discussion in the first section we can try to express the canonical bundle as a sum of an ample and an effective divisor. The G.I.T. construction gives us a large collection of ample divisors.

For our purposes we need only the following fact:

Lemma 3.1. *The divisor class λ is big and NEF.*

Proof. The shortest proof of this result is based on some facts about the Torelli map and the moduli spaces of abelian varieties. We can map the moduli space of curves $\overline{\mathrm{M}}_g$ to the moduli space A_g of principally polarized abelian varieties of dimension g by sending C to the pair $(Jac(C), \Theta)$ consisting of the Jacobian of C and the theta divisor. In characteristic zero this map extends from $\overline{\mathrm{M}}_g$ to the Satake compactification. The class λ is a multiple of the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$ from the embedding of A_g by theta constants. The lemma follows. □

A much more precise theorem due Cornalba and Harris [CH] determines the restriction of the ample cone of $\overline{\mathrm{M}}_g$ to the plane spanned by λ and δ.

Theorem 3.2. *Let a and b be any positive integers. Then the divisor class $a\lambda - b\delta$ is ample on $\overline{\mathrm{M}}_g$ if and only if $a > 11b$.*

For a nice exposition of the proof see [HM1] §6.D.

Remark 3.3. Note that λ itself is not ample, but since it is big it is a sum of an ample and an effective divisor. Consequently, it suffices to express the canonical bundle of $\overline{\mathrm{M}}_g$ as a sum of λ and an effective divisor.

4. The moduli space is of general type

In this section we would like to sketch the main steps of the proof of the following fundamental theorem due to Harris, Mumford and Eisenbud. You can read more about the details in [HM1] §6.F. The papers [HM2], [H] and [EH5] contain the proofs.

Theorem 4.1. *The moduli space of curves $\overline{M}_g$ is of general type if $g \geq 24$.*

The strategy of the proof is to show that the canonical class of the moduli space of curves is numerically equivalent to the sum of an ample and an effective divisor. We already know that the class of any divisor on the moduli space may be expressed as a linear combination of the classes λ and the boundary divisors δ_i.

We know that the canonical class of $\overline{\mathrm{M}}_g$ is given by the formula

$$K_{\overline{M}_g} = 13\lambda - 2\delta - \delta_1.$$

We also know that since $(11+\epsilon)\lambda - \delta$ is ample, λ is big. Hence if we could find an effective divisor

$$a\lambda - b_0\delta_{irr} - b_1\delta_1 - \cdots - b_{\lfloor g/2 \rfloor}\delta_{\lfloor g/2 \rfloor}$$

satisfying the inequalities

$$\frac{a}{b_i} < \frac{13}{2}, \quad \frac{a}{b_1} < \frac{13}{3}$$

then this will show that the canonical bundle is big because it may be expressed as the sum of a big and effective class.

There are two main difficulties with the approach we have outlined so far. First the construction of effective divisors with small slope is a difficult problem. We will see that the Brill-Noether and Petri divisors will do the job for Theorem 4.1. However, the calculation of these divisor classes are not easy.

The second problem is that even if we show that there are many canonical forms on $\overline{M}_g$, this does not necessarily prove that the moduli space is of general type. The problem is that $\overline{M}_g$ is singular. It is possible that canonical forms defined on the smooth locus do not extend to a desingularization. In fact, this is not the case. All the singularities of $\overline{M}_g$ are canonical, hence the canonical forms defined on the smooth locus extend to any desingularization. More precisely:

Theorem 4.2. *Let $g \geq 4$. Then for every n, the n-canonical forms defined on the locus of curves without automorphisms extend to n-canonical forms on a desingularization of $\overline{M}_g$.*

A sketch of some ideas. We will briefly outline some of the main ideas that go into the proof. For a complete argument see [HM2].

The proof relies on Reid-Tai Criterion. Let G be a finite group acting on a finite dimensional vector space V linearly. Let V^0 be the locus where the action is free. The Reid-Tai criterion answers the question of when pluri-canonical forms extend from V^0/G to a desingularization of V/G. For all $g \in G$, let g be conjugate to a matrix $Diag(\zeta^{a_1}, \ldots, \zeta^{a_d})$ where ζ is a primitive mth root of unity and $0 \leq a_i < m$. If for all $g \in G$ and ζ

$$\sum_{i=1}^{d} \frac{a_i}{m} \geq 1$$

then any pluri-canonical form on V^0/G extends holomorphically to a desingularization of V/G.

In view of the Reid-Tai Criterion one has to check whether $\sum_{i=1}^{d} \frac{a_i}{m} \geq 1$ holds and in cases it does not hold verify by hand that the pluri-canonical sections extend holomorphically to a desingularization. The following theorem characterizes the stable curves that fail to satisfy the Reid-Tai criterion.

Theorem 4.3. *Let C be a stable curve of arithmetic genus $g \geq 4$. Let ϕ be an automorphism of C of order n. Let ζ be a primitive n-th root of unity and suppose that the action of ϕ on $H^0(\Omega_C \otimes \omega_C)$ is given by $Diag(\zeta^{a_1}, \ldots, \zeta^{a_{3g-3}})$. Then one of the following possibilities hold:*

(1) $\sum_{i=1}^{3g-3} \frac{a_i}{m} \geq 1$.

(2) *C is the union of an elliptic or one-nodal rational curve C_1 meeting a curve C_2 of genus $g-1$ at one point. ϕ is the hyperelliptic involution on C_1 and the identity on C_2.*

(3) *C is the union of the elliptic curve C_1 with j invariant 0 meeting a curve C_2 of genus $g-1$ at one point. ϕ is an order 6 automorphism of C_1 and is the identity on C_2.*

(4) *C is the union of the elliptic curve C_1 with j invariant 12^3 meeting a curve C_2 of genus $g-1$ at one point. ϕ is an order 4 automorphism of C_1 and is the identity on C_2.*

The proof of this result rests on a case by case analysis of the possibilities based on a lemma that solves the problem for smooth curves.

Lemma 4.4. *Let C be a smooth curve. Let ϕ be an automorphism of C of order n. Let ζ be a primitive n-th root of unity and suppose that the action of ϕ on $H^0(\Omega_C \otimes \omega_C)$ is given by $Diag(\zeta^{a_1}, \ldots, \zeta^{a_{3g-3}})$. Then one of the following possibilities hold:*

(1) $\sum_{i=1}^{3g-3} \frac{a_i}{m} \geq 1$.

(2) *C is a genus zero or one curve.*

(3) *C is a hyperelliptic curve of genus 2 or 3 and ϕ is the hyperelliptic involution.*

(4) *C is a genus 2 curve which is the double cover of an elliptic curve and ϕ is the involution exchanging the branches.*

The proof of the lemma is based on an analysis of the possibilities using the Riemann-Hurwitz formula.

The final step of the proof is to check by explicit computation that pluri-canonical forms extend to the resolution of the singularities over the loci that do not satisfy the Reid-Tai Criterion.

□

The fact that $\overline{\mathrm{M}}_g$ has canonical singularities allows us to carry out the naive program outlined above. We need effective divisors of small slope. The locus of curves that admit a degree d map to $\mathbb{P}^r$ where g, r, d satisfy the equality

$$g - (r+1)(g-d+r) = -1$$

form a divisor on $\overline{\mathrm{M}}_g$ called the Brill-Noether divisor. Its class is given by the following theorem:

Theorem 4.5. *If $g+1 = (r+1)(g-d+r)$, then the class of the Brill-Noether divisor on $\overline{M}_g$ is given by*

$$c\left((g+3)\lambda - \frac{g+1}{6}\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i\right)$$

where c is a positive rational constant.

Unfortunately this divisor exists only when $g+1$ is composite. When g is composite and $g+1$ is not, every curve admits finitely many degree d maps to $\mathbb{P}^r$ where

$$g - (r+1)(g-d+r) = 0.$$

The number of such maps may be determined by Schubert calculus. We can then try to define a divisor by asking that some of these maps not be distinct. This will essentially be the Petri divisor (we will give a more precise definition below).

Example 4.6. The Petri divisors in $g = 4$ and 6 are fun to describe. Consider a smooth, non-hyperelliptic curve C of genus 4. The canonical model of such a curve is the complete intersection in $\mathbb{P}^3$ of a quadric and a cubic surface. Such a curve lies on a unique quadric surface. If the quadric is a smooth quadric surface, then C possesses two (distinct) g^1_3s. They are given by projection to either of the factors of $\mathbb{P}^1 \times \mathbb{P}^1$. In codimension one C lies on a quadric cone. For such curves the two g^1_3s come together. The Petri divisor is simply the closure of such curves.

Exercise 4.7. Calculate the class of the divisor given by the closure of curves whose canonical model lies in a singular quadric.

Let C be a smooth, non-hyperelliptic curve of genus 6. A general such curve C lies on a Del Pezzo surface of degree 5 and contains 5 distinct g^2_6s corresponding to the ways of blowing down D_5 to $\mathbb{P}^2$. If C lies on a Del Pezzo surface with double points, then these g^2_6s are no longer distinct. Again the Petri divisor is the closure of the locus of such curves.

The Petri divisor is defined as the closure of the union of codimension one loci in $\overline{\mathrm{M}}_g$ of curves which possess a linear series $V \subset H^0(C, L)$ of degree d and dimension 1 such that the multiplication map

$$V \otimes H^0(C, K \otimes L^{-1}) \to H^0(C, K)$$

is not injective.

Theorem 4.8. *Let $g = 2(d-1)$. Then the class of the Petri divisor is given by*

$$\frac{2(2d-4)!}{d!(d-2)!}\left((6d^2+d-6)\lambda - d(d-1)\delta_{irr} - (2d-3)(3d-2)\delta_1 - \cdots\right)$$

where the coefficients of the remaining boundary divisors are negative and larger in absolute value than that of δ_1 (at least when $d > 4$).

The Brill-Noether and Petri divisors give us the necessary divisors to conclude the proof of Theorem 4.1. When $g \geq 24$ and odd, we can use the Brill-Noether divisor with $r = 1$. The relevant ratio is that of λ and δ_0 and is equal to

$$6 + \frac{12}{g+1}.$$

When $g \geq 24$ this is less than 6.5, hence the canonical class of $\overline{\mathrm{M}}_g$ is big provided $g+1$ is not prime. The Brill-Noether divisors also take care of the cases $g = 24, 26$. When g is even and greater than or equal to 28, the Petri divisor works to give the conclusion.

We will spend the next section calculating the class of the Brill-Noether divisor. The class of the Petri divisor is harder to compute. You can find the computation in [EH5].

Remark 4.9. Recently G. Farkas has announced that $\overline{\mathrm{M}}_{22}$ and $\overline{\mathrm{M}}_{23}$ are also of general type. The strategy of his proof is the same. He constructs more elaborate effective divisors.

5. The computatation of the classes of Brill-Noether Divisors

5.1. **The Brill-Noether Theorem.** In this subsection we will discuss some of the basics of Brill-Noether theory and the theory of limit linear series. Eisenbud and Harris have developed this theory in order to prove theorems like the Brill-Noether or Gieseker-Petri theorems. We will describe their approach to some of these problems. The best places to start learning about the subject are Chapter 5 of [HM1] and [ACGH]. Other good references [GH], [EH2], [EH1], [EH3], [EH4], [KL2], [KL1] among others.

Brill-Noether theory asks the following fundamental question:

Question 5.1. When can a curve of genus g be represented in $\mathbb{P}^r$ as a non-degenerate curve of degree d?

There is an expected answer to this question. We are asking when does there exist a degree d line bundle on a curve C of genus g with at least an $r+1$-dimensional space of global sections? We can calculate the expected dimension of this locus in $Pic^d(C)$ as follows. Let us twist all the line bundles in $Pic^d(C)$ by $\mathcal{O}_C(np)$ for a sufficiently large n (large enough to kill h^1). Over $Pic^d(C)$ there is a map between the push-forward of the Poincare bundle and the trivial bundle of rank n given by evaluation at the point p. We are interested in the dimension of the locus where the evaluation map has kernel of dimension at least $r+1$. The expected codimension of the locus is given by $(r+1)(g-d+r)$.

The Brill-Noether number is defined as follows

$$\rho(g,r,d) = g - (r+1)(g-d+r).$$

By the discussion in the previous paragraph on a general curve of genus g, we expect there to be a g^r_d if and only if this number is non-negative.

Example 5.2. One learns very early in one's algebraic geometry career that every Riemann surface admits a non-constant meromorphic function. One then ask given a genus g Riemann surface S what is the smallest degree meromorphic function on S?

(1) If S has genus zero, then there are non-constant meromorphic functions of degree one, namely the Möbius transformations.

(2) If S has genus one or two, then the smallest degree non-constant meromorphic function has degree 2. For instance, in the case of genus 1, the Weiestrass p function is such a function.

(3) If S has genus 3, already the story becomes more complicated. If S is hyperelliptic, then it does admit a meromorphic funciton of degree 2. However, not all genus 3 curves are hyperelliptic. They do not admit meromorphic functions of degree 2. However, non-hyperelliptic curves of genus 3 can be realized as plane quartics in $\mathbb{P}^2$. Projecting the quartic from a point on the curve gives a meromorphic function of degree 3.

(4) If S is a non-hyperelliptic curve of genus 4, then its canonical image is the complete intersection of a quadric and a cubic in $\mathbb{P}^3$. By projecting to one of the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ or the base of the Hirzebruch surface F_2 (in case the quadric is singular), we obtain a map of degree 3 to $\mathbb{P}^1$.

(5) If S is a non-hyperelliptic and non-trigonal curve of genus 5, then it is the complete intersection of three quadric hypersurfaces in $\mathbb{P}^4$. Hence such a curve does not admit a map of degree 3 to $\mathbb{P}^1$ (Exercise: why?). Show that a such a curve does admit a map of degree 4 to $\mathbb{P}^1$. (Hint: The intersection of two quadrics is a Del Pezzo surface of degree 4. The map to $\mathbb{P}^2$ blowing down 5 disjoint exceptional curves presents the curve as a five-nodal sextic. Project from a node.)

(6) Show that a general curve of genus 6 does not admit a curve of degree 2 or 3 to $\mathbb{P}^1$, but does admit a map of degree 4. (Hint: The canonical image of a general curve of genus 6 lies on a degree 5 Del Pezzo surface in $\mathbb{P}^5$.)

(7) One can carry the analysis a little further. In fact the following is known.

Proposition 5.3. *Every Riemann surface of genus g admits a non-constant meromorphic function of degree* $\lfloor \frac{g+3}{2} \rfloor$. *Moreover, a general Riemann surface of genus g does not admit a non-constant meromorphic function of smaller degree.*

We say that a curve C of genus g has a g^r_d if there exists a line bundle L of degree d on C with $h^0(C, L) \geq r$. The Brill-Noether theorem asserts that a general curve has a g^r_d if and only if the Brill-Noether number $\rho(g, r, d)$ is non-negative. In fact, more is true. Let $W(C)^r_d$ be the locus of line bundles in $Pic_d(C)$ that have at least $r+1$-dimensional space of global sections. Then for a general C, the dimension of this locus is given by the Brill-Noether number.

Theorem 5.4 (Brill-Noether, Kempf, Kleiman-Laksov, Griffiths-Harris, Eisenbud-Harris)**.** *Let C be a general curve of genus g. Then the dimension of $W(C)^r_d$ is equal to the Brill-Noether number. In particular, there exists a g^r_d on C if and only if the Brill-Noether number is non-negative. Moreover, in case $\rho(g, r, d) = -1$, the closure of the locus of smooth curves that possess a g^r_d is a divisor in $\overline{M}_g$.*

Remark 5.5. Note that the previous proposition is a special case of the Brill-Noether theorem. If we take $r = 1$, then we see that the Brill-Noether number is non-negative if and only if $d \geq \lfloor \frac{g+3}{2} \rfloor$.

A sketch of the proof. The idea of the proof goes back to Castelnuovo. Let us consider a g-nodal rational curve and try to calculate the dimension of the space of g^r_ds on such a curve. If the dimension is correct, then we have a chance of deducing the theorem for general curves by specializing them to a g-nodal rational curve. A map of degree d to $\mathbb{P}^r$ (where $r < d$) on a g-nodal rational curve amounts to the same thing as the projection of a rational normal curve of degree d from a $\mathbb{P}^{d-r-1}$ that meets g specified secant lines. In other words we are asking for the dimension of the intersection of g Schubert cycles Σ_r in $\mathbb{G}(d-r-1, d)$. Had these cycles been general we could conclude that the dimension of the space of g^r_d on a g-nodal rational curve is

$$(d-r)(r+1) - gr.$$

I leave it to you to verify that this is equal to the Brill-Noether number.

There are a few problems with the previous idea. First, the Jacobian of a g-nodal curve is not compact, so the limits of g^r_ds on a general curve need not be g^r_ds. Second, more serious problem, is that the Schubert cycles Σ_r are not general Schubert cycles, hence their intersection need not be dimension theoretically transverse. We will completely circumvent the first problem and deal with the second in the meantime by specializing to g-cuspidal curves. In other words, we will make the Schubert cycles Σ_r be defined with respect to tangent lines to the rational normal curve. Note that the semi-stable reduction of such a curve is the normalization of the curve with g elliptic tails attached at the points that map to the cusps. In particular, the non-compactness issue disappears.

Theorem 5.6 (Eisenbud-Harris)**.** *Let $p_1, \ldots, p_m$ be distinct points on a rational normal curve of degree d in $\mathbb{P}^d$. Let $F_1, \ldots, F_m$ be the osculating flags to the rational normal curve defined at these points, respectively. Then Schubert varieties defined with respect to the flags F_i in the Grassmannian, if non-empty, intersect in the expected dimension.*

The proof of this theorem is based on a Plücker formula. Let $V \subset H^0(C, L)$ be a linear series of vector-space dimension $r+1$ on a genus g curve C. Let

$$0 \le \alpha_0(p) \le \alpha_1(p) \le \cdots \le \alpha_r(p)$$

be the ramification sequence of V at a point p of C. Let $R_i(p)$ be the orders of vanishing of sections in V at p. Recall that the ramification sequence index $\alpha_i(p)$ is defined to be $\alpha_i(p) = R_i(p) - i$. The sum of all the ramification indeces over all points of the curve C may be expressed only in terms of the dimension of V, degree of L and the genus of C as the following proposition indicates.

Proposition 5.7. *Let V be a linear series of degree d and vector-space dimension $r+1$ on a genus g curve. Then the sum of the ramification indices satisfy the following equality*

$$\sum_{j,p} \alpha_j(p) = (r+1)d + \frac{r(r+1)}{2}(2g-2).$$

Proof of Proposition. The Taylor expansions of order r of the sections in V gives a map to the bundle of r-jets of sections of L

$$\alpha : V \otimes \mathcal{O}_C \to P^r(L).$$

Taking the $r+1$st exterior power we get a map

$$\mathcal{O}_C \to \bigwedge^{r+1} P^r(L).$$

The formula claimed in the proposition arises from calculating the number of zeroes of this map in two different ways. First of all using the exact sequence that relates principal parts bundles

$$0 \to L \times K_C^m \to P^m(L) \to P^{m-1}(L) \to 0$$

we see inductively that

$$\bigwedge^{r+1} P^r(L) \cong L^{r+1} \otimes K_C^{\frac{r(r+1)}{2}}.$$

Therefore, the number of zeros is equal to

$$(r+1)d + \frac{r(r+1)}{2}(2g-2),$$

which is the right hand side of the claimed formula.

On the other hand, we can calculate the number of zeros in local coordinates. At each point $p \in C$ we choose the sections of V that vanish to order $i + \alpha_i(p)$ in terms of a local coordinate t. The order of zeros of the map is the smalles order of vanishing of any linear combination of the $(r+1) \times (r+1)$ minors of the matrix

$$\begin{pmatrix} t^{\alpha_0(p)} & t^{1+\alpha_1(p)} & t^{2+\alpha_2(p)} & \cdots \\ \alpha_0(p)t^{\alpha_0(p)-1} & (1+\alpha_1(p))t^{\alpha_1(p)} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

This order is precisely the left hand side of the formula in the proposition. □

In particular, when the genus is equal to zero we see that the total ramification is equal to $(r+1)(d-r)$. Since the total ramification may not exceed this number it is now easy to conclude the Eisenbud-Harris Theorem. □

Exercise 5.8. Check that for a map of a rational curve to have a ramification sequence $\alpha_0, \dots, \alpha_{r+1}$ at p is equivalent to asking the center of the projection to satisfy the Schubert condition of codimension equal to the sum of the ramification indeces with respect to the osculating flag to C at p. Express the class of the Schubert variety in terms of the ramification sequence.

Another central theorem of curve theory that is amenable to similar (but more difficult) techniques is the Gieseker-Petri Theorem.

Theorem 5.9 (Gieseker-Petri, Eisenbud-Harris, Lazarsfeld)**.** *Let C be a general curve. Let L be any line bundle on C. Then the multiplication map*

$$H^0(C, L) \otimes H^0(C, K \otimes L^{-1}) \to H^0(C, K)$$

is injective.

Suppose that there exists a g^r_d with negative Brill-Noether number. Using Riemann-Roch for curves, we see that

$$h^0(K - g^r_d) = h^0(g^r_d) - d + g - 1 = r + 1 - d + g - 1 = r - d + g.$$

Since the Brill-Noether number is negative, we must have $(r+1)(r-d+g) \geq g+1$. Hence the domain of the map $H^0(C, L) \otimes H^0(C, K \otimes L^{-1})$, where L is the line bundle giving the g^r_d has dimension at least $g + 1$. Consequently, the Petri map cannot be injective. We conclude that for a Gieseker-Petri general curve there does not exist a g^r_d if the Brill-Noether number is negative.

Remark 5.10. In general, the failure of the injectivity cannot be explained by dimension theoretic reasons alone. Consider a genus 4 curve with a canonical form with a single zero (necessarily of multiplicity 6). The Weierstrass sequence for such a point is given as follows:

$$h^0(3p) = 2, \quad h^0(5p) = 3, \quad h^0(6p) = 4.$$

Although the target and the domain vector spaces in $h^0(3p) \otimes h^0(3p) \to h^0(6p)$ have the same dimension, the multiplication map is not an isomorphism since it is not possible to get a section vanishing to order 5 by multiplying sections vanishing to order 3.

Unfortunately, most easy to manipulate curves are not general in the sense of Gieseker-Petri. For example, a k-gonal curve, that is a curve admitting a non-constant holomorphic map of degree k to $\mathbb{P}^1$, if k is small ($k < (g+3)/2$ compared to g) will not satisfy the Gieseker-Petri Theorem as observed by the above calculation.

5.2. **Limit linear series.** In this subsection we will briefly sketch the theory of limit linear series for curves of compact type developped by Eisenbud and Harris in order to study Brill-Noether theory. Since Joe has written very good accounts of the theory our treatment will be brief. One of the main uses of the theory is to describe the closure of Brill-Noether conditions on singular curves. For more details see [HM1] Chapter 5, [EH1], [EH3], [EH4], [EH2].

Definition 5.11. A curve is of *compact type* if its dual graph is a tree.

Proposition 5.12. *The following conditions on an at worst nodal curve C of genus are equivalent*

1. *C is of compact type.*
2. *The sum of the geometric genera of the components of C equals g.*
3. *The Jacobian of C is compact.*

Proof. If C is of compact type, then its dual graph is a tree. In particular, every irreducible component of C is smooth and any two components meet at most in one point. We can prove the equivalence of 1 and 2 by induction. If the dual graph of C has only one vertex, then the equivalence is obvious. Suppose the result is true for C whose dual graphs have at most k vertices. Take a leaf of the dual graph of C with $k+1$ vertices. If we remove the leaf, the remaining curve is a curve of compact type whose dual graph has k vertices. Hence the sum of the geometric genera of its components equals its genus. Since the component we removed is attached at one point using the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_{C_1 \cap C_2} \to 0$$

we see that

$$h^1(C, \mathcal{O}_C) = h^1(C_1, \mathcal{O}_{C_1}) + h^1(C_2, \mathcal{O}_{C_2}).$$

This completes the proof that 1 implies 2.

To see that 2 implies 1, we observe that by the same exact sequence that the genus of a curves is at least the sum of the genus of its components. If there is a loop, then by the exact sequence the genus of the curve formed by a loop is one more than the sum of its components.

To see the equivalence of these conditions with the condition that the Jacobian is compact, we need to study the group line bundles on a singular curve. Let $\nu : \tilde{C} \to C$ be the normalization of the curve C.

We have an exact sequence

$$0 \to \mathbb{C}^* \to (\mathbb{C}^*)^r \to \Gamma(C) \to \mathrm{Pic}(C) \to \mathrm{Pic}(\tilde{C}) \to 0,$$

where r is the number of irreducible components of C. Consequently, $J(C)$ is compact if and only if the number of points lying over the singular points of the curve is two less than twice the number of irreducible components. But the latter can only happen if and only if the dual graph of the curve is a tree. This proves the equivalence of the conditions. □

The importance of curves of compact type arises from the fact that one can develop a theory of limits of line bundles on such curves. In fact, one can develop such a theory on tree-like curves. A Deligne-Mumford stable curve is tree-like if after normalizing the curve at its non-separating nodes one obtains a curve of compact type. In other words, a tree-like curve differs from curves of compact type so that the irreducible components may have internal nodes.

The main difficulty. Suppose you have a one-parameter family of curves $\mathcal{X} \to B$ such that the total space of the family is smooth, all the fibers but the central fiber are smooth curves and the central fiber is a reducible nodal curve with smooth components. Given line bundle L on $\mathcal{X} - X_0$, we can always extend it to the total space. Since $\mathcal{X} - X_0$ is smooth, the line bundle L corresponds to a Cartier divisor on

$\mathcal{X} - X_0$. We can take the closure of this divisor in $\mathcal{X}$ to obtain a Cartier divisor on $\mathcal{X}$ (Note that here we use the smoothness of the total space). Since Cartier divisors correspond to line bundles, there is a corresponding line bundle $\tilde{L}$ extending L.

Unfortunately, the extension is not unique. This is the main technical difficulty of the subject. Suppose the central fiber $X_0 = Y \cup Z$. If we twist $\tilde{L}$ by $\mathcal{O}_{\mathcal{X}}(mY)$ or $\mathcal{O}_{\mathcal{X}}(mZ)$, we do not change the line bundle L on $\mathcal{X} - X_0$; however, we obtain a different line bundle on the total space.

Definition 5.13 (Limit linear series)**.** Let C be a curve of compact type. A *limit linear series* D of degree d and dimension r on C is a linear series $|V_Y|$ of degree d and dimension r on every irreducible component of C called the *aspect* of D on Y, such that for any two components Y and Z of C meeting at a node p the aspects V_Y and V_Z satisfy

$$a_i(V_Y, p) + a_{r-i}(V_Z, p) \geq d.$$

The limit linear series is *refined* if the following inequalities are equalities for every i. The limit linear series is *crude* if one inequality is strict.

Using the Plücker formulae one may generalize the Brill-Noether theorem to curves of compact type. In fact, to tree-like curves as follows:

Theorem 5.14. *Let C be a tree-like curve. Suppose the following about the irreducible components of Y:*

(1) *If the genus of Y is 1, then Y meets the rest of the curve in one point.*

(2) *If the genus of Y is 2, then Y meets the rest of the curve in one point which is not a Weierstrass point.*

(3) *If the genus of Y is three or more, then Y meets the rest of the curve at general points*

If $p_1, \ldots, p_r$ are general points of C or arbitrary smooth points on rational components of C, then for any ramification sequence at the points p_i, the dimension of the special linear series with the given ramification sequences at the points has the expected dimension.

Remark 5.15. For our purposes, the important corollary of the theorem is that if we consider the pull-back of the Brill-Noether divisor to $\overline{\mathrm{M}}_{0,n}$ and $\overline{\mathrm{M}}_{2,1}$ via the map that attaches g fixed elliptic curves at the marked points and the map that attaches a fixed genus $g-2$ curve, respectively, the pull-back to $\overline{\mathrm{M}}_{0,n}$ is zero while the pull-back to $\overline{\mathrm{M}}_{2,1}$ is supported on the Weierstrass divisor.

5.3. **Calculating the classes of the Brill-Noether divisors.** In this subsection we complete our discussion of the proof of Theorem 4.1 by calculating the class of the Brill-Noether divisors. For the rest of this section assume that the Brill-Noether divisor is expressed as follows in terms of the standard generators

$$a\lambda - b_0\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i \delta_i.$$

We calculate the class by pulling-back the Brill-Noether divisor to $\overline{\mathrm{M}}_{2,1}$ and $\overline{\mathrm{M}}_{0,g}$. Using the first pull-back we obtain the relations

$$a = 5b_1 - 2b_2 \quad \text{and} \quad b_{irr} = \frac{b_1}{2} - \frac{b_2}{6}.$$

Using the second pull-back, we obtain for $i > 1$ the relations

$$b_i = \frac{i(g-i)}{g-1} b_1.$$

Solving for all the coefficients in terms of b_1, we obtain the class of the Brill-Noether divisors upto a positive constant. (One can determine the constant, but we do not need this for proving Theorem 4.1.)

Theorem 5.16. *If $g + 1 = (r + 1)(g - d + r)$, then the class of the Brill-Noether divisor on $\overline{\mathrm{M}}_g$ is given by*

$$c\left((g+3)\lambda - \frac{g+1}{6}\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i\right)$$

where c is a positive rational constant.

To conclude the proof we need to obtain the claimed relations between the coefficients. First, consider the map

$$at_{g-2} : \overline{\mathrm{M}}_{2,1} \to \overline{\mathrm{M}}_g$$

obtained by attaching a fixed genus $g - 2$ curve with a marked point to curves of genus 2 with a marked point along their marked points. The theory of limit linear series shows that the pull-back of the Brill-Noether divisor is a multiple of the divisor W on $\overline{\mathrm{M}}_{2,1}$ obtained by taking the closure of the locus in $\mathrm{M}_{2,1}$ where the marked point is a Weierstrass point. The first set of relations are obtained by comparing the class of W and the pull-backs of the standard generators by at_{g-2}

Claim 5.17. *The class of the Weierstrass divisor W is given by*

$$W = 3\omega - \lambda - \delta_1,$$

where ω is the class of the relative dualizing sheaf on $M_{2,1}$.

The pull-back of λ by at_{g-2} is λ on $\overline{\mathrm{M}}_{2,1}$. Similarly the pull-backs of δ_{irr} and δ_1 by at_{g-2} are δ_{irr} and δ_1 on $\overline{\mathrm{M}}_{2,1}$, respectively. By adjunction the pull-back of δ_2 is $-\omega$. It follows that by pulling back the Brill-Noether divisor and using the claim we obtain the relation

$$a\lambda - b_{irr}\delta_{irr} - b_1\delta_1 - b_2\omega = c(3\omega - \lambda - \delta_1).$$

We thus see that $b_2 = 3c$. Next we use the relation

$$10\lambda = \delta_{irr} + 2\delta_1$$

to solve for the other coefficients to obtain the first set of relations.

To calculate the class of the Weierstrass divisor W, we note that a Weierstrass point is a ramification point of the canonical linear series. Using this one can exhibit W as the degenracy locus of a map between vector bundles.

Exercise 5.18. Carry this out and complete the calculation of the class of W. (Hint: See page 338-339 in [HM1]).

Next, consider the map

$$att : \overline{\mathrm{M}}_{0,g} \to \overline{\mathrm{M}}_g$$

obtained by attaching a fixed one pointed elliptic curve to the marked points. To obtain the required relations among the coefficients of the boundary we consider the pull-back of the Brill-Noether divisors by π. Since the Brill-Noether divisor is disjoint from the imape of att, the pull-back of the divisor to $\overline{\mathrm{M}}_{0,g}$ is zero.

We thus obtain the following relation among the coefficients:

$$a\ att^*\lambda - b_0\ att^*\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i\ att^*\delta_i = 0.$$

We have to calculate the pull-backs of the standard divisors by att. Clearly, λ and δ_{irr} pull-back to zero. The pull-backs of the divisors δ_i are the classes δ_i^0 on $\overline{\mathrm{M}}_{0,g}$ (where we place a 0 to remind ourselves that these are the divisors on $\overline{\mathrm{M}}_{0,g}$) provided $i > 1$. The image of att is contained in $\Delta_1 \subset \overline{\mathrm{M}}_g$, so the pull-back of δ_1 is the trickiest. To calculate its class, we take a one-parameter family of curves

$$\pi : C \to B$$

in $\overline{\mathrm{M}}_{0,g}$. We may assume that every member of the family has at most two components and that the total space of the family is smooth. Contracting the components with fewer sections (or either of the components when equal numbers of sections pass through both components), we obtain a $\mathbb{P}^1$ bundle with g sections

$$\tilde{\pi} : \tilde{C} \to B.$$

Since the classes of any two sections differ by a multiple of the fiber class, the difference of two section classes has self-intersection zero.

The pull-back of δ_1 by att is the push-forward to the sum of the squares of the sections σ_i in the original family to the base. The sections γ_i in the projective bundle and in the original family are related by

$$\tilde{\pi}_*(\sum \gamma_i^2) = \pi_*(\sum \sigma_i^2) + \sum_{i=2}^{\lfloor g/2 \rfloor} i\ \delta_i^0.$$

Using that

$$\gamma_i^2 + \gamma_j^2 = 2\gamma_i \cdot \gamma_j$$

we obtain the relation

$$\tilde{\pi}_*(\sum \gamma_i^2) = \sum_{i=2}^{\lfloor g/2 \rfloor} \frac{i(i-1)}{g-1} \delta_i^0.$$

Combining these relations we obtain that

$$att^*\delta_1 = \sum_{i=2}^{\lfloor g/2 \rfloor} -\frac{i(g-i)}{g-1} \delta_i^0.$$

The class of the Brill-Noether divisors (up to a constant multiple) follow from these calculations.

6. The ample and effective cones of the moduli space of curves

The proof that $\overline{\mathrm{M}}_g$ is of general type when $g > 24$ required us to know a two-dimensional slice of the ample cone of $\overline{\mathrm{M}}_g$. Combining this with our knowledge of some special effective divisors we could conclude the proof. One may ask the more detailed questions:

Question 6.1. In terms of the generators of the picard group $\lambda, \delta_1, \dots, \delta_{\lfloor g/2 \rfloor}$ what is the ample cone of $\overline{\mathrm{M}}_g$? What is the effective cone of $\overline{\mathrm{M}}_g$?

Almost nothing is known about the effective cone of $\overline{\mathrm{M}}_g$. Of course, everytime one writes down an effective divisor, one generates part of the effective cone. In recent years G. Farkas has spent a tremendous amount of effort to construct effective divisors on $\overline{\mathrm{M}}_g$. Despite these efforts our understanding of the effective cone of the effective cone of $\overline{\mathrm{M}}_g$ has progressed little beyond examples of effective divisors. On the other hand, there is a beautiful conjecture giving a complete description of the ample cone of $\overline{M}_{g,n}$.

Remark 6.2. There is however one exception to our ignorance about the effective cone. The effective cone of $\overline{\mathrm{M}}_{0,n}$ is difficult to describe. However, if we quotient $\overline{\mathrm{M}}_{0,n}$ by the action of the symmetric group on n letters it becomes very easy to see that the effective cone is equal to the cone spanned by the boundary divisors.

Exercise 6.3. Show that the effective cone of $\overline{\mathrm{M}}_{0,n}/\mathfrak{S}_n$ is the span of the boundary divisors as follows: Show that if D is effective, then the coefficient of Δ_2 has to be non-negative (Hint: Consider a fixed $\mathbb{P}^1$ with n-marked points. Let the last marked point vary keeping the rest fixed. Show that such curves cover an open subset of $\overline{\mathrm{M}}_{0,n}/\mathfrak{S}_n$ and only intersect Δ_2 among the boundary divisors.) Show that the coefficient of Δ_t has to be non-negative by induction on t. (Hint: Assume that the effective divisor does not contain any of the boundary divisors as a fixed component. Fix a reducible curve with $t-1$ points on one component and $n-t+1$ points on the other component. Attach the first component at a fixed point of the first curve to a variable point on the second curve. Considering this curve complete the induction.)

$\overline{M}_{g,n}$ has a stratification according to topological type. Given a curve C we can associate the dual graph to C. For every irreducible component we associate a vertex. For every node connecting two irreducible components we associate an edge between the two vertices. For every self node we associate a loop based at the vertex corresponding to the irreducible component. Finally, we associate a tail emanating from the appropriate vertex for each marked point. We label the vertices by the geometric genus of the corresponding curve.

We obtain a stratification of $\overline{M}_{g,n}$ by considering the loci of curves with a fixed dual graph. The codimension of a stratum is equal to the number of nodes that the curve represented by that graph has. The zero dimensional strata consist of curves with $3g-3+n$ nodes. We proved that every component of such a curve is a $\mathbb{P}^1$ whose normalization contains exactly three distinguished points. The one dimensional strata consist of curves with $3g-4+n$ nodes. Every component but one of a curve with $3g-4+n$ nodes is a $\mathbb{P}^1$ whose normalization has three distinguished points. The remaining component is either a $\mathbb{P}^1$ whose normalization has four distinguished points or a genus one curve whose normalization has one distinguished point. We can view the one-dimensional loci as the images of $\overline{\mathrm{M}}_{0,4}$ and $\overline{\mathrm{M}}_{1,1}$. These curves are often referred to as F-curves.

The F-conjecture describes the ample cone of $\overline{M}_{g,n}$ in terms of the F-curves.

Conjecture 6.4 (The F-conjecture)**.** *A divisor on $\overline{M}_{g,n}$ is ample if and only if it intersects positively with every F-curve.*

Of course, by Kleiman's criterion every ample divisor intersects every curve positively. The content of the F-conjecture is to say that checking for the F-curves suffices. Alternatively the conjecture may be formulated as saying that the Mori cone of curves on $\overline{M}_{g,n}$ is generated by the F-curves.

Observe that from this statement one can obtain very explicit inequalities describing the ample cone of $\overline{M}_{g,n}$. For simplicity we will give the description when $n = 0$.

Exercise 6.5. Determine inequalities describing the ample cone of $\overline{M}_{g,n}$ in terms of the generators of the Picard group when $n \geq 1$.

We begin by enumerating the F-curves. As already observed every component but one of a curve parameterized by a general point on an F-curve corresponds to a $\mathbb{P}^1$ with 3 distinguished points. If the remaining component is a genus 1 curve with one marked point, then when we separate the curve at this marked point we obtain a curve of genus $g-1$ consisting of $\mathbb{P}^1$s with three distinguised points each and the genus one curve with one marked point. This curve is obtained by attaching a fixed curve with one marked point to $\overline{M}_{1,1}$. From this it follows that $C \cdot \delta_i = 0$ for $i \geq 1$. To calculate $C \cdot \delta_0 = 1/2$, we observe that Finally, $C \cdot \lambda = 1$.

If the remaining component is a genus 0 curve with 4 marked points, then the normalization restricted to that component might be injective. In this case if we split the curve at the distinguisehd points, we obtain four pieces of genus g_1, g_2, g_3 and $g - g_1 - g_2 - g_3$.

Exercise 6.6. Work out the intersections of the F-curves with the boundary components. Use these intersections to give inequalities that describe an upper bound on the ample cone.

Problem 6.7. Show that every divisor in the cone dual to the F-curves is ample (or give a counterexample).

Currently the F-conjecture is open. A. Gibney has verified the conjecture for $\overline{\mathrm{M}}_g$ for many small genera. There is also a general result due to Gibney, Keel and Morrison [GKM] that reduces the general conjecture to the case of genus zero:

Theorem 6.8. *The F conjecture holds for $\overline{M}_{g,n}$ if it holds for $\overline{M}_{0,m}$ for $m \leq g+n$.*

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