Lecture 22: Bertini's Theorem, Coherent Sheves on Curves

Let's consider some ways to construct smooth varieties.

Theorem 1.1 (Bertini's Theorem)**.** *Let $X \subseteq \mathbb{P}V$ be a smooth subvariety. Then for a generic hyperplane H, $Y = X \cap H$ is again smooth.*

Recall that the set of hyperplanes is parametrized by the dual projective space $\mathbb{P}V^\vee$. To say that a hyperplane is *generic* is equivalent to saying that there is a nonempty open subset $U \subseteq \mathbb{P}V^\vee$ containing the point in $\mathbb{P}V^\vee$ corresponding to that hyperplane and such that each hyperplane in U possesses the desired property.

Proof. We can assume that X is irreducible. Indeed, if X has multiple irreducible components (i.e. is not connected) and if we know the claim for each irreducible component, then we have a finite set of open subsets in $\mathbb{P}V^\vee$, whose intersection is again open and consists of hyperplanes whose intersection with X is smooth.

Let $d = \dim(X)$, $n = \dim(\mathbb{P}V)$. For all $x \in X$, we have T_xX of dimension d, and $T_xX \subseteq T_x\mathbb{P}V$. If $x \in H$, then $H \cap X$ will be smooth at x if $T_xH \not\supset T_xX$. Consider the following subset $Z \overset{\text{def}}{=} \{(H, x) \mid H \ni x, T_xH \supset T_xX\}$ of the product $\mathbb{P}V^\vee \times X$. One easily sees that is is closed. The set of H for which $H \cap X$ is singular is the image of Z under the projection $\mathbb{P}V^\vee \times X \to \mathbb{P}V^\vee$.

We will now proceed by dimension count. First, we want to calculate the dimension of Z. For this, consider the projection $Z \to X$. The two conditions from the definition of Z clearly say that if $(H, x) \in Z$, then H contains a subspace W of dimension d isomorphic to $\mathbb{P}^d$, so the fiber at each point is $\{H \in \mathbb{P}V^\vee \mid H \supset W\} = \mathbb{P}(V/W)^\vee$. Since $\dim(V) = n + 1$, $\dim(W) = d + 1$, we have the fiber isomorphic to $\mathbb{P}^{n-d-1}$. Recall from a theorem last time that a generic fiber has dimension equal to the difference of the dimensions of the two spaces, so $\dim(Z) = n - 1$.

If we let $\pi : Z \to \mathbb{P}V^\vee$, then $\overline{\pi(Z)}$ has dimension at most $n - 1$, so the complement $\mathbb{P}V^\vee \setminus \overline{\pi(Z)}$ is not empty. Moreover, this complement is exactly the desired open subset, and this concludes the proof. □

Corollary 1. *A generic hypersurface of degree d is smooth. Moreover, if $X \subset \mathbb{P}^n$ is smooth, for a generic hypersurface S of degree d, $S \cap X$ is smooth.*

Proof. Use Veronese embedding, consider $\mathbb{P}^n \subset \mathbb{P}^N$ where $(t_1, \ldots, t_n) \to (t^I)$ where I ranges over all monomials of degree d. Then a hypersurface becomes a hyperplane in this case, then we reduce to the previous case. □

Remark 1. *Assume that X is irreducible of dimension d. If X is not contained in a hyperplane H, then we know that each component of $X \cap H$ has dimension $d - 1$. If X is projective, then $X \cap H$ is nonempty. In fact, one can check that if $\dim(X) > 1$ and H is a general hyperplane, then $X \cap H$ is irreducible.*

Remark 2. *Bertini's theorem refers to a range of theorems. For instance, we can allow X to be singular, and one of the variations of Bertini's theorems will say something about the singularities of $X \cap H$.*

Remark 3. *We can also relate the topology of X and that of $X \cap H$ — this is called the Lefschetz Hyperplane Theorem. For instance, the map $H^i(X, \mathbb{C}) \to H^i(X \cap H, \mathbb{C})$ is an isomorphism up to the middle degree for a general hyperplane H.*

Coherent Sheaves on Curves Now we start the last main topic — the sheaf cohomology. We will mostly focus on the case of sheaves on curves.

Let $\mathcal{F}$ be a coherent sheaf on a smooth irreducible curve.

Definition 1. *The* torsion subsheaf *$\mathcal{T} \subseteq \mathcal{F}$ is a subsheaf of $\mathcal{F}$ generated by torsion sections.*

The torsion subsheaf $\mathcal{T}$ has finite support (by Noetherian property and due to the dimension equal to one), and $\mathcal{F}/\mathcal{T}$ is a torsion free sheaf. But we know that a finitely-generated torsion free module over a DVR is free, so a torsion free sheaf is locally free. Moreover, $0 \to \mathcal{T} \to \mathcal{F} \to \mathcal{F}/\mathcal{T} \to 0$ splits noncanonically by constructing a surjection $\mathcal{F} \to \mathcal{T}$; this follows from the corresponding result about modules over DVRs. It follows that a coherent sheaf $\mathcal{F}$ on a curve can be decomposed into a direct sum $\mathcal{T} \oplus \mathcal{F}'$, where the first summand is a torsion sheaf and the second one is torsion-free.

Every torsion sheaf $\mathcal{T}$ has finite length. If its support is irreducible, then it is just a point, so in this case $\mathcal{T} \cong \mathcal{O}_x$ for some x. Actually, a torsion sheaf has a filtration with gr $T = \bigoplus \mathcal{O}_{x_i}$. In fact, this result is true for a torsion sheaf on any variety X if the sheaf has finite support.

Now let $\mathcal{E}$ be a locally free sheaf, and $\mathcal{E}' \subset \mathcal{E}$ be a subsheaf. Of course, if $\mathcal{E}$ is torsion-free, then so is $\mathcal{E}'$. However, this is not the case for $\mathcal{E}/\mathcal{E}'$. Consider the following example where we have torsion in the quotient:

$$0 \to \mathcal{O}(-x) \to \mathcal{O} \to \mathcal{O}_x \to 0.$$

Another example is when we can take $X = \operatorname{Spec}(k[t])$, and consider $\mathcal{O} \xrightarrow{t} \mathcal{O}$.

Locally we have $\mathcal{E} = \mathcal{O}^{\oplus r}, \mathcal{E}' = \mathcal{O}^{\oplus r'}$, then $\mathcal{E}' \to \mathcal{E}$ can be given by a $r' \cdot r$ matrix with entries in $\mathcal{O}$.

Exercise 1. *Using Nakayama lemma, show that the quotient has torsion at x if and only if evaluating matrix coefficients at x gives us a matrix of rank less than r'.*

We want to call a subbundle such a locally free sheaf that taking quotient with respect to it gives a locally free sheaf.

Example 1. *For example, if $r' = 1$, this just means sections can vanish at that point. Consider $\mathcal{O} \to \mathcal{O}^{\oplus r}$, given by $(f_1, \ldots, f_r)$, then cokernel has torsion at x iff $f_i(x) = 0$ for all i. Recall that $f_i \in \mathcal{O}_{x,X}$, and this holds if the valuation of each f_i is greater than 0. If d is the minimum of these valuations, and t is some element of $\mathcal{O}_{x,X}$ with valuation 1 (i.e. $t \in \mathfrak{m}_x - \mathfrak{m}_x^2$), then we have $\mathcal{O} \xrightarrow{t^d} \mathcal{O} \xrightarrow{f_i/t^d} \mathcal{O}^r$ which is the same as the map above. The second map has no cotorsion (i.e. torsion in the cokernel), and the image is independent of the choices.*

In general, for $\mathcal{E}' \subset \mathcal{E}$, there exists unique $\mathcal{E}''$, such that $\mathcal{E}' \hookrightarrow \mathcal{E}'' \hookrightarrow \mathcal{E}$ where the second map has no cotorsion, and the rank of $\mathcal{E}''$ is the same as rank of $\mathcal{E}'$ i.e. $\mathcal{E}''/\mathcal{E}'$ is torsion. To construct such a sheaf $\mathcal{E}''$, we first take the torsion subsheaf $\mathcal{T} \subset \mathcal{E}/\mathcal{E}'$ and then consider its preimage with respect to the surjection $\mathcal{E} \to \mathcal{E}/\mathcal{E}'$. The latter will be the desired $\mathcal{E}''$, as one can easily verify.

Definition 2. *We call $\mathcal{E}''$ the saturation of $\mathcal{E}'$ in $\mathcal{E}$.*

Basic invariants of a coherent sheaf: rank and degree

Definition 3. *Let $\mathcal{F}$ be a coherent sheaf. The rank of $\mathcal{F}$ is defined as the rank of the locally free sheaf* ($\mathcal{F}$/torsion) *when we work over smooth varieties. More generically (for any irreducible variety), one defines* rank *as follows. For a field $K \overset{def}{=} \varinjlim_U k[U]$, we have the following K-vector space: $V_{\mathcal{F}} \overset{def}{=} \varinjlim_U \mathcal{F}[U]$. The rank is the dimension $\operatorname{rk}(\mathcal{F}) \overset{def}{=} \dim_K(V_{\mathcal{F}})$.*

One can show that rank is equal to the dimension of a generic fiber of $\mathcal{F}$.

It is clear from the definition that rank is additive in short exact sequences.

Definition 4. *$K^0(\mathcal{A})$, the Grothendieck group of an abelian category $\mathcal{A}$, is the free abelian group generated by isomorphism classes in $\mathcal{A}$ modulo the relation that, given $0 \to A \to B \to C \to 0$, we have $[B] = [A] + [C]$.*

This is the universal object for invariants that are additive in short exact sequences. Thus for instance rank is a homomorphism $K^0(\operatorname{Coh}(X)) \to \mathbb{Z}$. Note that $K^0(\operatorname{Coh}(X))$ can be explicitly described for X of dimension one.

Assume now that X is complete. Define another homomorphism $\delta : K^0(\operatorname{Coh}(X)) \to \mathbb{Z}$ such that $\delta([\mathcal{E}]) \mapsto \deg(\det(\mathcal{E}))$ where $\mathcal{E}$ is locally free. Additivity comes from multiplicativity of the determinant in short exact sequences. For torsion sheaves, we set δ to be the length of $\mathcal{T}$, which is the same as the dimension of $\Gamma(\mathcal{T})$. (Recall that the length ℓ is defined as the number of summands in gr $\mathcal{T} = \bigoplus_{i=1}^{\ell} \mathcal{O}_{x_i}$.)

This would make sense. Consider the short exact sequence $0 \to \mathcal{O} \to \mathcal{O}(D) \to \mathcal{O}_D \to 0$. The first sheaf has degree 0, the second second one has degree $\deg(D)$, whereas the leftmost has length $\deg(D)$. But we still need a formal check.

Proposition 1. *δ is a well-defined homomorphism.*

Lemma 1. *If we have a short exact sequence $0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{T} \to 0$, where $\mathcal{T}$ is torsion and the other two sheaves are torsion free, then $\deg(\mathcal{E}') = \deg(\mathcal{E}) + \ell(\mathcal{T})$.*

Proof. Induction on $\ell(\mathcal{T})$, reduce to $\mathcal{T} = \mathcal{O}_x$, and $r = \operatorname{rank}(\mathcal{E}) = \operatorname{rank}(\mathcal{E}')$. We claim that $\Lambda^r(\mathcal{E}) \to \Lambda^r(\mathcal{E}')$ has a zero of order 1 at x. Locally it looks like $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & t \end{pmatrix}$ where $t \in \mathfrak{m} - \mathfrak{m}^2$. □

Proof of the Proposition. We have $\delta(\mathcal{E} \oplus \mathcal{T}) = \deg(\det(\mathcal{E})) + \ell(\mathcal{T})$. Need to check that for $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, we have the additive property. First consider $0 \to \mathcal{T}' \to \mathcal{T} \to \mathcal{T}/\mathcal{T}' \subseteq \mathcal{T}'' \to 0$, then we have $\delta(\mathcal{T}) = \delta(\mathcal{T}') + \delta(\mathcal{T}/\mathcal{T}')$ also $\delta(\mathcal{F}) = \delta(\mathcal{F}/\mathcal{T}) + \delta(\mathcal{T})$ and same for $\mathcal{F}', \mathcal{F}''$, so we reduce to the case where $\mathcal{F} = \mathcal{E}$ is torsion free. If $\mathcal{F}'_s$ is the saturation of $\mathcal{F}'$, then $\delta(\mathcal{F}'_s) = \delta(\mathcal{F}') + \delta(\text{torsionof } \mathcal{F}'')$, so replacing $\mathcal{F}'$ by $\mathcal{F}'_s$ doesn't check the RHS of $\delta(\mathcal{F}) + \delta(\mathcal{F}') + \delta(\mathcal{F}'')$, so we can check all three of them to locally free, which we have already discussed above. □

Remark 4. *The homomorphism δ can be refined to a homomorphism $K^0(\mathrm{Coh}(X)) \to \mathrm{Pic}(X)$ followed by the degree map $\mathrm{Pic}(X) \to \mathbb{Z}$.*

Cohomology of quasicoherent sheaves Cohomology is an important invariant of quasicoherent sheaves. To cut a long story short, cohomology of a sheaf is the derived functor of the global sections. Some theory can be found in Grothendieck's Tohoku paper, which is worth reading. A derived functor accounts for the nonexactness of the initial functor between abelian categories.

Definition 5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. A* δ-functor *is a collection of functors $F^i : \mathcal{A} \to \mathcal{B}$ such that for every short exact sequence $0 \to A \to B \to C \to 0$ we have a long exact sequence $0 \to F(A) \to F(B) \to F(C) \to F^1(A) \to F^1(B) \to F^1(C) \to F^2(A) \to \dots$ that is functorial in short exact sequences.*

Definition 6. *A δ-functor is* universal *if it has a canonical morphism from any δ-functor. In other words, it is the terminal object in the category of δ-functors.*

Definition 7. *The universal δ-functor is called the* derived functor*, and is of course unique if exists. We denote it by $\mathcal{R}^i F$.*

In our case, $\mathcal{A} = \mathrm{QCoh}(X), \mathcal{B} = \mathbf{Vect}$, $F = \Gamma$.

Next class we'll show the existence along with some properties, including Serre duality for curves.