Lecture 19: Smoothness, Canonical Bundles, the Adjunction Formula

Last time we defined Ω_X, T_xX and smoothness. We proved that any X contains an open dense smooth subset, that X is smooth at x if and only if Ω_X is locally free around x, and X is smooth if and only if Ω_X is locally free.

Here's a trivial observation: suppose we have a surjection $f : A \to B$ and $\mathfrak{m}_B \in B$ an maximal ideal, let $\mathfrak{m}_A = f^{-1}\mathfrak{m}_B$, then $\mathfrak{m}_B/\mathfrak{m}_B^2 = \mathfrak{m}_A/\mathfrak{m}_A^2 + I$ where $I = \ker(f)$. If $Y = \operatorname{Spec} B$ contains $X = \operatorname{Spec} A$, $y \in X \subseteq Y$, then $T_y^*Y = T_y^*X/(df_i)$, where f_i are generators of I.

Corollary 1. *We have the following:*

1. *If $X \subseteq \mathbb{A}^n$ is a hypersurface given by the equation $I_X = (P)$, then $x \in X$ is smooth if and only if $dP|_x \in T_x^*\mathbb{A}^n \neq 0$, i.e. $\left.\frac{\partial P}{\partial x_i}\right|_x \neq 0$ for some i.*
2. *Suppose $X \subseteq \mathbb{A}^n$ has dimension $n-m$ where $I_X = (f_1, \dots, f_m)$ (this is not true for all X), then X is smooth at a point x if and only if $df_i|_x \in T_x^*\mathbb{A}^n = k^n$ are linearly independent, i.e. the m-by-m matrix $\left(\left.\frac{\partial f_i}{\partial x_j}\right|_x\right)$ has rank m.*

Proof. The first claim is a particular case of the second. $\dim(X) \geq \dim_x X \geq n-m \implies \dim_x X = n-m$. Now apply the definition of smoothness, and that $T_x^*X = T^*\mathbb{A}^n/(df_i|_x)$. $\square$

If $X \subseteq \mathbb{P}^m$ has dimension $n-m$, $I_X = (F_1, \dots, F_m)$ for homogeneous polynomials, then $x = (x_0, \dots, x_n)$ is a smooth point if and only if $\left(\left.\frac{\partial f_i}{\partial x_j}\right|_x\right)$ has rank m. To see this, note that X is smooth at x if and only if C_X (the cone) is smooth at $\tilde{x}$ because C_X is locally isomorphic to $X \times \mathbb{A}^1$, and note that $T^*_{(x,y)}(X \times Y) = T_x^*X \oplus T_y^*Y$.

Proposition 1. *Suppose $X \subseteq \mathbb{A}^n$, $x \in X$ is a smooth point if and only if $\exists f_1, \dots f_m \in I_X \subseteq k[x_1, \dots, x_n]$ which locally generate I_X and $df_i|_x$ are linearly independent.*

Proof. If $f_1, \dots, f_n$ as above exists, then $\dim(T_x^*X) = n-m$ while $\dim_x X \geq n-m$ also $\dim_x X \leq (\dim T_x^*X) \implies \dim_x X = \dim T_x^*X$ i.e. x is a smooth point. Conversely, suppose X is smooth at x, pick $f_1, \dots, f_m \in I_X$ such that $f_i|_x$ form a basis in $\ker(T_x^*\mathbb{A}^n \to T_x^*X)$. Then by the first part of the proof, $Z = (f_1, \dots, f_m)$ is smooth at x with $\dim_x Z = n-m = \dim_x X$, where $Z \supseteq X$. So we are done if we know that Z is locally irreducible, which follows from the next lemma:

Lemma 1. *$\widehat{\mathcal{O}_{Z,x}} = \varprojlim_n k[Z]/\mathfrak{m}_x^n \cong k[[t_1, \dots, t_{n-m}]]$ (i.e. is a free ring).*

Why does this imply Z locally irreducible? Z locally irreducible means $\mathcal{O}_{Z,x}$ has no zero divisors, which would follow from the fact that $\mathcal{O}_{Z,x} \subseteq \widehat{\mathcal{O}_{Z,x}}$ which follows from Nakayama. In particular, $\ker(\mathcal{O}_{Z,x} \to \widehat{\mathcal{O}_{Z,x}}) = \bigcap_n \mathfrak{m}_x^n$ which is a finitely generated ideal $\mathcal{O}_{Z,x}$ is Noetherian, and we have $\mathfrak{m}_x I = I \implies I = 0$.

Remark 1. *This lemma is equivalent to that $\bigoplus_n \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ (the associated graded ring) is isomorphic to $k[t_1, \dots, t_n]$. The general case is given in the next lemma.*

Lemma 2. *Let A be a ring, $\mathfrak{m}$ a maximal ideal, $a \in A$. Suppose $a \in \mathfrak{m}^p$, write $\overline{a} \in \overline{A}_p = \mathfrak{m}^p/\mathfrak{m}^{p+1}$ and $\overline{A} = \bigoplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Then $\overline{A/(a)} = \bigoplus (A/a)^n/(A/a)^{n+1} = \overline{A}/(\overline{a})$ if $\overline{a}$ is not a zero divisor.*

Proof. $(\overline{A/(a)})_n = \mathfrak{m}^n/(\mathfrak{m}^{n+1} + (aA \cap \mathfrak{m}^n))$, $\overline{A}/(\overline{a}) = \mathfrak{m}^n/\mathfrak{m}^{n+1} + a\mathfrak{m}^{n-p}$. For any $x \in \mathfrak{m}^k$, we have $ax \in \mathfrak{m}^{k+p}$; if $\overline{a}$ is not a zero divisor, then $x \notin \mathfrak{m}^{k+1}$, then $ax \notin \mathfrak{m}^{k+p+1}$. $\square$

Now we return to the first lemma. $f_1, \dots, f_n$ have linearly independent differential at x, by induction check $\widehat{k[x_1, \dots, x_n]/(f_1, \dots, f_i)} = k[[t_1, \dots, t_{n-1}]]$, i.e. $\mathrm{gr}(k[x_1, \dots, x_n]/(f_1, \dots, f_i)) \cong k[t_1, \dots, t_{n-1}]$, and if so, wlog can assume $\overline{f_{i+1}} = t_1$.

Proposition 2. *X is smooth at x iff $\widehat{\mathcal{O}_{X,x}} \cong k[[t_1, \dots, t_d]]$ where $d = \dim_x X$.*

Proof. The forward direction follows from the proof of the previous proposition where we deduced this from the fact that X is locally given by equations with independent differentials. For the other direction, assume $\widehat{O}_{X,x} \cong k[[t_1, \dots, t_d]]$ then we want to conclude $d = \dim T_x^* X$. It suffices to check that $\dim_x X \geq d$. Pick $f_1, \dots, f_d \in \mathfrak{m}_x$ with linearly independent differentials, and we claim that $(f_1, \dots, f_d)$ is a regular sequence, i.e. f_{i+1} is not a zero divisor in $\mathcal{O}_{X,x}/(f_1, \dots, f_i)$. Then $f_{i+1} \neq 0$ on each component of $Z_{f_1,\dots,f_i}$ passing through x, so we get $X \supsetneq Z_1 \supsetneq Z_2 \dots \supsetneq Z_d \ni x$, where Z_i is a component in $Z_{f_1,\dots,f_i}$. Why is it a regular sequence? because $\widehat{\mathcal{O}_{X,x}/(f_1, \dots, f_i)} \cong k[[t_1, \dots, t_{m-i}]] \supseteq \mathcal{O}_{X,x}/(f_1, \dots, f_i)$ (check by induction). $\square$

This concludes the proof of the proposition. $\square$

Proposition 3. *Suppose $Z \subseteq X$ is a closed subvariety.*

1. *We have an exact sequence $\mathscr{I}_Z/\mathscr{I}_Z^2 = \mathscr{I}_Z|_Z \to \Omega_X|_Z \to \Omega_Z \to 0$ (recall $\mathscr{F}|_Z = \mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathscr{F} = \mathscr{F}/\mathscr{I}_Z\mathscr{F}$ allows us to identify a sheaf $\mathscr{F}$ on Z with $i_*\mathscr{F}$).*

2. *If for all $x \in Z$, $\mathscr{I}_Z$ is locally (around x) generated by $f_1, \dots, f_m$ such that $df_i|_x$ are linearly independent at x, then the sequence is short exact, and $\mathscr{I}_Z/\mathscr{I}_Z^2$ is a locally free sheaf of rank m where m is the codimension.*

In the situation of (2), $\mathscr{I}_Z/\mathscr{I}_Z^2$ is called the *conormal* bundle.

Example 1. *X, Z are smooth irreducible, $\dim(Z) = \dim(X) - 1$, $Z = D$ is a divisor, $\mathscr{I}_Z = \mathcal{O}(-D)$ is an invertible sheaf. A local section of it is a function vanishing on Z. We can send f to a 1-form df vanishing on D, and it defines a section of the conormal bundle.*

Definition 1. *If X is a smooth irreducible variety of dimension d, then $\Omega(X)$ is a locally free sheaf of rank d. Then the top exterior power $\omega(X) = \bigwedge^d \Omega(X)$ is a locally free sheaf of rank 1. We call it the canonical line bundle or the canonical sheaf ("canonical" because any smooth variety gets it for free).*

If $0 \to A \to B \to C \to 0$ is a short exact sequence of locally free sheaves, then we have

$$\bigwedge^{top}(B) = \bigwedge^{top}(C) \otimes \bigwedge^{top}(A).$$

Corollary 2 (Adjunction Formula)**.** $\omega_D = \omega_X(-D)|_D$.

One last comment: the graded algebra has a nice geometric property as follows:

Definition 2. $\mathrm{Spec}(\mathrm{gr}(\mathcal{O}_{X,x}))$ *is called the tangent cone to X at x.*

Proposition 4. *The tangent cone is the cone over the exceptional locus in the blowup at x.*