Lecture 14: (Quasi)coherent sheaves on Projective Spaces

First an abstract lemma. Let $\mathcal{L} : \mathcal{C}_1 \to \mathcal{C}_2, \mathcal{R} : \mathcal{C}_2 \to \mathcal{C}_1$ be an adjoint pair; if $\mathcal{L}$ is fully faithful and $\mathcal{R}$ is conservative, then they are inverses. The unit is $Id \xrightarrow{u} \mathcal{L} \circ \mathcal{R}$ and the counit is $Id \xrightarrow{\varepsilon} \mathcal{R} \circ \mathcal{L}$. Additionally, we have $\mathcal{R} \xrightarrow{\varepsilon(\mathcal{R})} \mathcal{R} \circ \mathcal{L} \circ \mathcal{R} \xrightarrow{\mathcal{R}(u)} \mathcal{R} = Id$.

Example 1. $\mathcal{C}_1 = \mathcal{C}_2 =$ ***Vect***. *Let V be a finite dimensional vector space. Let $\mathcal{R} : U \to V \otimes U$, $\mathcal{L} : U \to V^* \otimes U = \operatorname{Hom}(V, U)$. Then the operation above becomes $V \xrightarrow{\delta \mapsto Id \otimes \delta} V \otimes V^* \otimes V \xrightarrow{E \otimes \delta \mapsto E(\delta)} V$.*

$\mathcal{L}$ is fully faithful implies $Id \cong \mathcal{R} \circ \mathcal{L}$. What about $\mathcal{L} \circ \mathcal{R} \cong Id$? It suffices to use $\mathcal{R} \to \mathcal{R} \circ \mathcal{L} \circ \mathcal{R} \to \mathcal{R}$. Last time we showed that the set of affine maps between X and Y is the same as the set of quasicoherent sheaves of $\mathcal{O}_X$-algebras (which are locally finitely generated and reduced).

Definition 1. *$X \to Y$ is a vector bundle if locally $\cong \mathbb{A}^n \times Y$, i.e. there exists a covering $f^{-1}(U_i) \cong \mathbb{A}^n \times U_i$ and agree on the intersection, i.e. the two copies of $\mathbb{A}^n \times (U_i \cap U_j)$ are glued together using $GL_n(k[U_i \cap U_j])$.*

The equivalence between the category of locally free sheaves and the category of vector bundles is given by $\mathcal{E} \mapsto \operatorname{Spec}(\oplus_i \operatorname{Sym}^i(\mathcal{E}))$, which is a contravariant functor. The opposite maps are from a vector bundle to the sheaf of sections of the dual bundle. Note that the total space is given by $\operatorname{Tot}(\mathcal{E}) = \operatorname{Spec}(\operatorname{Sym}(\mathcal{E}^\vee))$ where $\mathcal{E}^\vee = \operatorname{Hom}(\mathcal{E}, \mathcal{O})$.

We know that quasicoherent sheaves over an affine variety correspond to the modules over its coordinate ring. What about projective varieties? For a graded module M, define a quasicoherent sheaf on $\mathbb{P}^n$, denoted $\tilde{M}_{\mathbb{P}^n}$, as follows: its section on U is $\left(\tilde{M}_{\mathbb{A}^{n+1}}(\tilde{U})\right)_0$, where $\tilde{U}$ is the lifting of U to the cone $\mathbb{A}^{n+1} - \{0\}$. Say if $\mathbb{P}^n \setminus U = Z_f$, f is a degree d homogeneous polynomial, then $\tilde{M} = \varinjlim \frac{1}{f^i} \tilde{M}_{di}$ (again this is formal symbol).

Proposition 1. *The following are true:*

1. *$M \mapsto \tilde{M}_{\mathbb{P}^n}$ is an exact functor.*
2. *Every $\mathcal{F}$ that is a quasicoherent sheaf on $\mathbb{P}^n$ is of the form $\tilde{M}$ for some M, every coherent such $\mathcal{F}$ comes from some finitely generated M.*

Moreover, given a quasicoherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, $\mathcal{F} \cong \tilde{M}$ where $M = \bigoplus_{n \geq 0} \Gamma(\mathcal{F}(n))$.

Remark 1. *$M \to \tilde{M}_{\mathbb{P}^n}$ is not an equivalence. If M is finite dimensional, then $\tilde{M} = 0$. Also, $\tilde{M}_{\mathbb{P}^n}$ depends on the grading. For instance, if $M = A$ (a finite dimensional polynomial ring) is the standard grading, then $\tilde{M} = 0$; but if we use the shifted grading $M = A[i]$, i.e. $M_d = A_{i+d}$, then $\tilde{M} = \mathcal{O}(i)$.*

Proof. We have $\mathcal{F} \in \mathbf{QCoh}(\mathbb{P}^n)$, $\mathbb{A}^{n+1} - \{0\} \xrightarrow{j} \mathbb{A}^{n+1}$ and also $\mathbb{A}^{n+1} \xrightarrow{\pi} \mathbb{P}^n$. Exercise: $\pi_* \pi^* \mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}(n)$. On the other hand, $j_* \pi^*(\mathcal{F})$ is a quasicoherent sheaf on $\mathbb{A}^{n+1}$, and its global sections are the same as that of $\pi^* \mathcal{F}$, which is the same as that of $\pi_* \pi^*(\mathcal{F})$, which is $\bigoplus_{n \in \mathbb{Z}} \Gamma(\mathcal{F}(n))$. Let this be denoted M', which contains $M = \bigoplus_{n \geq 0} \Gamma(\mathcal{F}(n))$, and M'/M is concentrated on negative degrees, then we see that $\widetilde{M'/M}_{\mathbb{P}^n} = 0$, thus $\widetilde{M'}_{\mathbb{P}^n} = \tilde{M}_{\mathbb{P}^n}$. On the other hand, $\widetilde{M'}_{\mathbb{A}^n} = j_* \pi^* \mathcal{F}$, $\widetilde{M'}_{\mathbb{P}^n}(U) = j_* \pi^*(\mathcal{F}(\tilde{U}))_0 = \pi^*(\mathcal{F})(\tilde{U})_0 = \mathcal{F}(U)$. Now suppose the sheaf is coherent. Then $\mathcal{F} = \tilde{M}_{\mathbb{P}^n}$ for some M, $M = \bigcup M^i$, where each M^i is a finitely generated module, then $\mathcal{F} = \bigcup \widetilde{M^i}$. $\mathcal{F}$ being coherent implies $\mathcal{F} = \widetilde{M^i}$ for some i. □

Corollary 1. *If $\mathcal{F}$ is coherent, then there exists d, k, such that $\mathcal{O}(-d)^{\oplus k} \to \mathcal{F}$ is a surjection (equivalently, a surjection $\mathcal{O}^{\oplus k} \to \mathcal{F}(d)$). In other words, every coherent sheaf is a quotient of a vector bundle.*

Proof. If $\mathcal{F} = \tilde{M}$, M finitely generated, pick $d \geq$ degrees of all generators, it follows then that $M_{\geq d}$ is generated by M_d. But then $\tilde{M}_{\geq d} = \tilde{M}$. On the other hand, by definition of being finitely generated, we have $A^{\oplus k}[-d] \to M$ surjective, and then we have $\mathcal{O}^{\oplus k}(-d) \to \tilde{M}$ surjective. □

We have checked that the map $\mathbf{Mod}_{gr}(A) \to \mathbf{QCoh}(\mathbb{P}^n)$ and $\mathbf{Mod}_{gr,f.g.}(A) \to \mathbf{Coh}(\mathbb{P}^n)$ are both exact surjective on isomorphism classes and both kill some objects. In the second case, $\tilde{M} = 0$ iff M is finitely dimensional; in the first case, $\tilde{M} = 0$ iff M is locally nilpotent, i.e. for every x there exists some d such that $t_i^d x = 0$ for every i.

Serre Subcategory Given an abelian category A, a *Serre subcategory* is a full subcategory closed under extension. If B is a Serre subcategory, then one can define a new Serre quotient category A/B, universal among categories with a functor from A sending B to 0.

Proposition 2. *$\mathbf{QCoh}(\mathbb{P}^n)$ is equivalent to $\mathbf{Mod}_{gr}(A)$ mod out the locally nilpotent elements, and $\mathbf{Coh}(\mathbb{P}^n)$ is equivalent to $\mathbf{Mod}_{gr,f.g.}(A)$ mod out the finite dimensional elements.*

Proof. More generally, suppose $U \subseteq X$ is open, and $X \setminus U = Z$, we show that $\mathbf{QCoh}(U) = \mathbf{QCoh}(X)/\{\mathcal{F} \mid \operatorname{Supp}(\mathcal{F}) \subseteq Z\}$. The same holds for coherent sheaves. (To get the statement above, take $X = \mathbb{A}^{n+1}$, $Z = \{0\}$, $U = \mathbb{A}^{n+1} - \{0\}$.) Recall that A-module M is the same as a quasicoherent sheaf on X. A graded A-module M, on the other hand, corresponds to a quasicoherent sheaf that is equivariant with respect to the multiplicative group G_m action by definition, where $G_m = \operatorname{Spec}(k[t, t^{-1}]) \cong \mathbb{A}^1 - \{0\}$. Then $\mathbb{P}^n = (\mathbb{A}^{n+1} - 0)/G_m$, thus $\mathbf{QCoh}(\mathbb{P}^n) = \mathbf{QCoh}^{G_m}(\mathbb{A}^{n+1} - 0) = \mathbf{QCoh}^{G_m}(\mathbb{A}^{n+1})/(\mathcal{F} \text{ suchthat } \operatorname{Supp}(\mathcal{F}) \subseteq Z)$. $\square$

Internal Hom and tensor product of quasicoherent sheaves If we have $\mathcal{F}, \mathcal{G}$ quasicoherent, define the *internal hom* $\underline{\operatorname{Hom}}_{\mathbf{QCoh}(U)}(\mathcal{F}, \mathcal{G})(U) = \operatorname{Hom}(\mathcal{F}(U), \mathcal{G}(U))$, then obviously this is a sheaf of $\mathcal{O}$-modules. If $\mathcal{F}$ is coherent, then this is quasicoherent. $\mathcal{F} \otimes \mathcal{G}$ is the sheafification of the presheaf given by section-wise tensor product, and is a quasicohrerent sheaf. In particular, note if X is affine, we have $\tilde{M} \otimes_{\mathcal{O}} \tilde{N} = \widetilde{M \otimes_A N}$.

Invertible Sheaves If $\mathcal{F}$ is a locally free of rank 1 (a.k.a. an invertible sheaf), $\mathcal{F} \otimes \mathcal{G}$ is locally isomorphic to $\mathcal{G}$. Example: $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$. Why are they called invertible? if $\mathcal{F}$ is locally free of rank n, form $\mathcal{F}^\vee = \underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{O})$, then $\mathcal{F}^{\vee\vee} = \mathcal{F}$, and $\underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{G}) = \mathcal{F}^\vee \otimes_{\mathcal{O}} \mathcal{G}$. Now if $\mathcal{F} = \mathcal{L}$ is locally free of rank 1, then $\mathcal{L}^\vee \otimes \mathcal{L} = \underline{\operatorname{Hom}}(\mathcal{L}, \mathcal{L}) = \mathcal{O}$. Additionally, if $\mathcal{L}_1, \mathcal{L}_2$ are rank 1 locally free, then their tensor product is again locally free of rank 1. And obviously, $\mathcal{O} \otimes \mathcal{F} = \mathcal{F}$.

Corollary 2. *Isomorphism classes of invertible sheaves on X is an abelian group under tensor product.*

This is known as the Picard group $\operatorname{Pic}(X)$. Now let's describe it. For now, let X be irreducible.

Definition 2. *The Weil divisor group $DW(X)$ is a free abelian group spanned by irreducible codimension 1 subvarieties.*

A typical element in there has the form $D = \sum_i n_i D_i$ where $n_i \in \mathbb{Z}$, and D_i are the said subvarieties. If all the $n_i \geq 0$, then D is called *effective*.

Definition 3. *The Cartier divisor group $DC(X) = \Gamma(\mathcal{K}^*/\mathcal{O}^*)$, where $*$ means nonzero, and $\mathcal{K}$ is the sheaf of rational functions. Another way to describe it is the set of invertible fractional ideals. It can be seen as a subsheaf realized in $\mathcal{K}^*$.*

Theorem 1.1. *When X is factorial (for instance, when X is smooth), $DW(X) = DC(X)$. Generally, $\operatorname{Pic}(X) = DC(X)/K^*$, i.e. the quotient of Cartier divisors by the principal divisors.*

We'll see next time that $\operatorname{Pic}(\mathbb{P}^n) = \mathbb{Z} = \{\mathcal{O}(d)\}$.

Example 2. *Using invertible sheaf to embed a variety X in $\mathbb{P}^n$. In particular, $X = \mathbb{P}^1$. Let $\mathcal{L} = \mathcal{O}(n)$, where $n \geq 1$, $V = H^0(\mathcal{O}(n)) = \operatorname{Sym}^n(k \oplus k)$ (of dimension $(n+1)$), then we get a map from $\mathbb{P}^1$ to the projectivization of this space, which is $\mathbb{P}^n$. The image of this emdedding corresponds to degree n polynomials that are nth power of linear polynomials.*

MIT OpenCourseWare
http://ocw.mit.edu

18.725 Algebraic Geometry
Fall 2015

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.