October 28, 2003
NO LECTURE 11/13

1 Fibers of morphisms

Let $W \subset Y$ and let $f : X \to Y$. Then we have a commutative diagram:

$$\begin{array}{ccc} f^{-1}W & \hookrightarrow & X \\ \downarrow & & \downarrow f \\ W & \hookrightarrow & Y \end{array}$$

Then we can picture $f^{-1}(W)$ as a subset of the product $X \times W$, since there is a map to both W and X.

Ex. Blow-up of $\mathbb{A}^n$ at the origin.

Consider $\mathbb{A}^n \times \mathbb{P}^{n-1} \xrightarrow{p_1} \mathbb{A}^n$ and consider $X = \{(\underline{x}, [\underline{y}]) | x_i y_j = x_j y_i \forall i, j\}$. To show that X is closed, we will show that $X \cap \mathbb{A}^n \times U_{y_j \neq 0}$ is closed. Write $z_i = y_i/y_j$. Then the coordinate ring of $\mathbb{A}^n \times U_{y_j \neq 0}$ is $k[x_1, \dots, x_n] \otimes k[z_0, \dots, \hat{z_j}, \dots, z_n] = k[\underline{x}, \underline{z}]$. The equations in the definition of X are $x_i(y_l/y_j) = x_l(y_i/y_j)$ or $x_i z_l = x_l z_i$, so this is closed since it is defined by a polynomial.

Let $b : X \to \mathbb{A}^n$ by inclusion and then composition with the map onto the $\mathbb{A}^n$ component. We will compute the fibers of b. That is, compute $b^{-1}(a_1, \dots, a_n)$. Then this would correspond to a point $(a_1, \dots, a_n) \times [y_1 : \dots : y_n]$ where $y_j = (a_j/a_i)y_i$ if some $a_i \neq 0$. Then $y_i \neq 0$ and so WLOG, $y_i = 1$, so then we get $[a_0/a_i : \dots : a_n/a_i]$. Thus, the fiber is a single point! (Note, if $a_i = 0$ for all i, the fiber is NOT finite: we get $0 \times \mathbb{P}^{n-1}$.) So we get $b^{-1}(\mathbb{A}^n - \{0\}) \to \mathbb{A}^n - \{0\}$.

Ex. Consider $X = \{(t, x, y) | xy = t\}$. Then $X \subset \mathbb{A}^3$ and $g : X \to \mathbb{A}^1$ where $g : (t, x, y) \mapsto t$. What are the fibers? $F_a = g^{-1}(a)$ then for all a, $F_a = V(xy - a)$. If $a \neq 0$ then this is irreducible, but if $a = 0$ then $F_a = V(xy)$ which is not irreducible.

Ex. $y^2 = x^2(x + 1)$. What are the fibers under projection onto the x-coordinate? I zoned out. Something interesting that comes up is that this graph looks like an α. Two interesting points are the vertical tangent on the left and the point of intersection: the fibers there are different.

1.1 General picture

Reduce to morphisms $f : X \to Y$ with $f(X) \subset Y$ dense.

Def. f is *dominant* if $\overline{f(X)} = Y$.

Prop. $f : X \to Y$ a morphism, set $Z = \overline{f(X)}$. Then Z is irreducible and $X \xrightarrow{f} Z \hookrightarrow Y$ where the map $X \to Z$ is dominant.

Pf. Suppose $Z = Z_1 \cup Z_2$ where $Z_i \subset Z$ closed. Let $X = f^{-1}(Z_1) \cup f^{-1}(Z_2)$. Since X is irreducible, it must only be contained in one of these, so WLOG, $X \subset f^{-1}(Z_1)$. Thus, $f(X) \subset Z_1$ and so $Z = \overline{f(X)} \subset Z_1$. That this is dominant is obvious.

Remark. If f is dominant, then there is a natural inclusion $k(Y) \hookrightarrow k(X)$. Recall $k(Y) = \lim_{\emptyset \neq U \subset Y} \Gamma(U, O_Y) \to \lim_U \Gamma(f^{-1}(U), O_X) \subset \lim_{\emptyset \neq V \subset X} \Gamma(V, O_X) = k(X)$.

Theorem. Let $f : X \to Y$ be dominant, $W \subset Y$ irreducible and closed, and Z an irreducible component of $f^{-1}(W)$ that dominates W. Then $\dim Z \geq \dim W + r(*)$ where $r = \dim X - \dim Y$.

Pf. We can assume Y is affine. Let s be the codimension of W in Y. Now $(*) \iff \text{codim}_X Z \leq s$, because

$$\dim X - \dim Z \leq \dim X - \dim W - \dim X - \dim Y = \dim Y - \dim W.$$

If Y is affine there exist $g_1, \ldots, g_s \in \Gamma(Y, O_Y)$ such that W is a component of $V(g_1, \ldots, g_s)$. Let $f_i \in \Gamma(X, O_X)$ be $f^*(g_i)$. We have $Z \subset V(f_1, \ldots, f_s) \subset X$. Let $Z' \subset V(f_1, \ldots, f_s)$ be an irreducible component containing Z. We have $W = \overline{f(Z)} \subset \overline{f(Z')} \subset V(g_1, \ldots, g_s)$. Since W is an irreducible component of $V(g_1, \ldots, g_s)$ then we must get $W = \overline{f(Z')}$. But now, Z' is just Z as we've defined Z. Thus, Z is a component of $V(f_1, \ldots, f_s)$ so the codimension of Z in X is at most s, as desired.

Cor. If Z is a component of $f^{-1}(y)$ for $y \in Y$, then $\dim Z \geq r$.

Thm. If $f : X \to Y$ is dominant, $r = \dim X - \dim Y$ then there is a non-empty open $U \subset Y$ such that

1. $U \subset f(X)$
2. For all irreducible closed $W \subset Y$ such that $W \cap U \neq \emptyset$ and components Z of $f^{-1}(W)$ such that $Z \cap f^{-1}(U) \neq \emptyset$, $\dim Z = \dim W + r$.

pf. Next time. Noether normalization. In fact, we'll show that after replacing Y and X by dense opens we get a factorization $X \to Y \times \mathbb{A}^r \to Y$ where the map from X is finite.

Def. Let X be a variety, $S \subset X$ is called *constructible* if it is a finite union of locally closed subsets of X.

Thm. (Chevalley) If $f : X \to Y$ is a morphism, then $f(X) \subset Y$ is constructible.

Ex. (Not locally closed, but constructible). Consider $\mathbb{A}^2 - \{(0, y) | y \neq 0\}$.

Pf. of Chevalley: Induction on dimension of Y. If f is not dominant then let $Z = \overline{f(X)}$, then Z has lower dimension and consider $X \to Z$. If f is dominant, then by the previous theorem, there are $U \subset Y$ dense opens such that $f^{-1}(U) \twoheadrightarrow U$, let $Y - U = Z_1 \cup \ldots \cup Z_l$ since $Y - U$ is closed. We then get W_{ij} components of $f^{-1}(Z_i)$, and $f(X) = U \cup \cup_i f(W_{ij})$, which is constructible by induction.