October 21, 2003.

1 HW 5 problem

Thm. If R and S are integral domains and k-algebras, then $R \otimes_k S$ is an integral domain.

Pf. It is enough to consider R and S finitely generated, since we can write $R = \cup_i R_i$ and $S = \cup_j S_j$ so $R \otimes S = \cup_{i,j} R_i \otimes S_j$.

Say $f, g \in R \otimes S$ not zero but $fg = 0$. Then $f, g \in R_i \otimes S_j$ for some i, j.

Consider $k[x_1, \ldots, x_n] \twoheadrightarrow R$ and $k[y_1, \ldots, y_m] \twoheadrightarrow S$. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$, and consider

$J \hookrightarrow k[x_1, \ldots, x_n, y_1, \ldots, y_m] \twoheadrightarrow R \otimes S$.

We will prove that J is a prime ideal. To do this, we consider $Z \subset \mathbb{A}^{n+m}$ the closed algebraic set defined by J. We will prove that Z is irreducible, and then that $\sqrt{J} = J$.

To show that Z is irreducible, suppose $Z = Z_1 \cup Z_2$ where $Z_i \subset Z$ are closed. We have maps of topological spaces $Z \xrightarrow{p_1} X$ and $Z \xrightarrow{p_2} Y$. Observe: if $x \in X$ then $p_1^{-1}(x) \to Y$ is a homeomorphism, because $p^{-1}(x)$ is the set of quotients $R \otimes S \to k$ which factor through $R \otimes S \xrightarrow{x \otimes 1} S$.

So, $p_1^{-1}(x)$ is irreducible. Also, however, $p_1^{-1}(x) = (Z_1 \cap p_1^{-1}(x)) \cup (Z_2 \cap p_1^{-1}(x))$ so we know $p_1^{-1}(x) \subset Z_i$ for some i.

Define $X_i = \{x \in X | p_1^{-1}(x) \subset Z_i\}$. We know $X = X_1 \cup X_2$. If we can prove that $X_i \subset X$ are closed, then $X = X_{i_0}$ for either $i_0 = 1$ or 2, so $Z = Z_{i_0}$.

To prove that X_i is closed, consider for every $y \in Y$, the set $X_i(y) = p_1(Z_i \cap p_2^{-1}(y)) = \{x \in X | (x, y) \in Z_i\}$. Note that $X_i = \cap_y X_i(y)$. Also note that X_i is the preimage of Z_i under the map $x \mapsto (x, y)$, so X_i is closed, and therefore X_i is closed.

Now to prove that $\sqrt{J} = J$, we just need to show there aren't any nilpotent elements in $R \otimes S$. Say $h = \sum f_i \otimes g_i$ is nilpotent in $R \otimes S$ where $\{f_i\}$ and $\{g_i\}$ are linearly independent. For every $x \in X$ we get some $\sum f_i(x) g_i \in S$ which is nilpotent, which cannot be because S is an integral domain. Thus, $f_i(x) = 0$ for every $x \in X$. Then, $h = 0$ because $f_i = 0$ for every i.

We use that k is algebraically closed in order to map $R \otimes S \xrightarrow{x \otimes 1} S$ because this is essentially a quotient of R by a maximal ideal, which we only know is k if k is algebraically closed (or it could be some algebraic extension of k.)

2 Dimension

If X is a variety, then $\dim(X) = \text{tr.deg}_k k(X)$. Last time we showed that if $Z \subset X$ is irreducible, proper, and closed, then $\dim(Z) < \dim(X)$.

Thm. If X is a variety, $g \in \Gamma(X, O_X)$ non-zero, let Z be an irreducible component of $\{x \in X | g(x) = 0\}$. Then $\dim(Z) = \dim(X) - 1$. So for example, a hypersurface in $\mathbb{P}^n$ (a variety defined by a single polynomial) has dimension $n - 1$.

It suffices to consider X affine. We write $R = \Gamma(X, O_X)$.

The irreducible components of $V(g)$ correspond to the minimal primes $P \subset R$ that contain g.

Thm. (Krull's principal ideal theroem): If R is an integral domain, finitely generated k-algebra, $g \in R$ not zero, and P is a minimal prime containing g, then $\text{tr.deg}_k R/P = \text{tr.deg}_k(R) - 1$.

Pf. of KPIT. We want to reduce to the case $X = \mathbb{A}^n$.

Recall the Noether Normalization lemma: if X is an affine variety of dimension n, then there is a finite surjective morphism $\pi : X \to \mathbb{A}^n$. Recall that *finite* here means that $\Gamma(X, O_X)$ is a finitely generated $k[x_1, \ldots, x_n]$-module. This all matches up with our previous way of stating Noether normalization.

Lemma. Let $f : X \to Y$ be a finite morphism. Then

1. f is a closed map
2. Fibers are finite (from HW)
3. f is surjective $\iff$ $S = \Gamma(Y, O_Y) \to R = \Gamma(X, O_X)$ is injective.

Pf. of lemma. (2) was from HW. (1) and (3). Say we have $V(A) \subset X$ What is $f(V(A)) \subset Y$? It is the set of maximal ideals $f^{*-1}(m)$ where $m \subset R$ is maximal and contains A.

Let $f^{*-1}(A) = B$ be an ideal in S. We claim: $f(V(A)) = V(B)$. Note we have $S/B \hookrightarrow R/A$ integral. By the going-up theorem, for every maximal ideal $n \subset S/B$ there is an $m \subset R/A$ such that $m \cap S/B = n$. This shows that f is a closed map.

Now, if we take A to be the zero ideal, then $f(X) = V(Ker(S \to R))$. This proves (3), because if f is surjective then $f(X) = Y$ so the kernel is 0, and if the kernel is 0 then $V(0) = Y$ so f is surjective. This completes the proof of the lemma.

Now back to our proof of KPIT. We reduce to the case $P = \sqrt{(g)}$, which gets rid of all the components of Z except one. We can write

$$\sqrt{(g)} = P \cap P_1' \cap \ldots \cap P_t'.$$

Choose $f \in P_1' \cap \ldots \cap P_t'$ such that $f \notin P$ (by the prime avoidance theorem), and consider $D(f) \subset X$. Localize at f, and we will be left with the case $\sqrt{(g)} = P$.

Remark. If R is a UFD then we're done. This is because $g = eh^l$ where h is irreducible and $P = (h)$ and e a unit. Thus we want the transcendence degree of $R/(h)$ which is clearly $\text{tr.deg } R - 1$.

For the general case, choose a finite map $S = k[x_1, \ldots, x_n] \to R$. Let $L = \text{Frac }(S)$ and let $K = \text{Frac }(R)$. Let $g_0 = \text{Norm}_{K/L}(g)$. Recall that $\text{Norm}_{K/L} = \det(*g : K \to K)$ where we think of $*g$ the map (which multiplies by g) as a matrix. Then $g_0 \in S \cap P$. This is not obvious, so look it up.

Then we have $\text{tr.deg } R/P = \text{tr.deg } S/(S \cap P)$. Clain: $S \cap P = \sqrt{(g_0)}$. We have that $P \cap S \supset \sqrt{(g_0)}$. For the other way, say that $h \in P \cap S$. Then $h \in P$, so $h^l = fg$ in R. Now we take the norm of both sides and get $h^{l[K:L]}$ on the left, and $\text{Norm }(f) g_0$ on the right since the determinant is multiplicative, so a power of h is in (g_0). Now we can apply the same argument from when we had R as a UFD, so this completes the proof of KPIT.

3 Next stuff

Def. Let X be a variety, $Z \subset X$ closed. Then Z has *pure (co-)dimension* r if all the irreducible components of Z have (co-)dimension r.

Cor. If $g \in \Gamma(X, O_X)$ non-zero then $V(g) \subset X$ has pure co-dim 1.

Remark. Here is the converse. Let $Z \subset X$ be irreducible with codim 1 and a nonzero $g \in \Gamma(X, O_X)$ such that $g(Z) = 0$. Then Z is an irreducible component of $V(g)$ (as opposed to a subset of an irreducible component).

Cor. Let X be a variety, $Z \subsetneq X$ a maximal irreducible closed subset. Then $\dim Z = \dim X - 1$.

Pf. We can assume X is affine, so Z is associated with some ideal $(f_1, \ldots, f_l) \subset \Gamma(X, O_X)$, and note that $Z \subset V(f_1)$. So Z is equal to an irreducible component of $V(f_1)$ so $\dim Z = \dim X - 1$.

Cor. Say $\emptyset \neq Z_1 \subsetneq Z_2 \ldots \subsetneq Z_r \subsetneq X$ is a maximal length chain of irreducible closed subsets. Then $\dim X = r$. (By induction.)