

Goal:

1. Products exist in the category of prevarieties.
2. Product of two affine varieties is an affine variety.
3. Product of two projective varieties is a projective variety.

We did the affine case:

$$X \times Y \iff \Gamma(X, O_X) \otimes_k \Gamma(Y, O_Y).$$

We proved, if X is an affine variety, Y any prevariety, then there is an isomorphism between $\text{Hom}(Y, X) \cong \text{Hom}_{k\text{-Alg}}(\Gamma(X, O_X), \Gamma(Y, O_Y))$.

Furthermore, if the product exists, then as a set it must be $X \times Y$ (consider $\text{Hom}((*, k), X) = \{\text{points of } X\}$.)

Let $|Z|$ be the set $X \times Y$. We want a topology on $|Z|$ and a sheaf of k -valued functions O_Z such that for all affine opens $U \subset X, V \subset Y$, the set $U \times V$ is open and $O_Z|_{U \times V} = O_{U \times V}$. Thus, $U \times V$ will be our open covering of Z by affine sets.

Last time, we showed that if we can do this, then $(|Z|, O_Z)$ is the product in the category of prevarieties (i.e., we checked the universal property).

Define a set $E \subset |Z|$ to be open if for every affine open $U \subset X$ and $V \subset Y$, $E \cap (U \times V) \subset (U \times V)$ is open as a subset of $U \times V$ (which we know is a variety, and thus already has a variety.) This gives a topology on $|Z|$. We need to check that this is the same topology we get if X and Y are affine.

We need to check that $U \times V$ is open in Z and the topology on $U \times V$ is the induced topology.

Lemma. Let $U' \subset U$ be an affine open subset of an affine open set, then the topology $U' \times V$ is that induced by $U' \times V \hookrightarrow U \times V$.

Pf. $R' = \Gamma(U', O_{U'}), R = \Gamma(U, O_U), S = \Gamma(V, O_V)$. We know that $R \otimes S \rightarrow R' \otimes S$. A basis for the topology on $U' \times V$ is $D(\sum f_i \otimes g_i), f_i \in R', g_i \in S$. Write $U' = \cup_i D(h_i), h_i \in R$.

So $U' \times V = \cup_i D(h_i \otimes 1)$ where $h_i \otimes 1 \in R \otimes S$. So, $U' \times V \subset U \times V$ is open, and $D(\sum_i f_i \otimes g_i) = \cup_i D(h_i^N \otimes 1)(\sum_i f_i \otimes g_i)$, where N is some value (whatever is necessary).

Lemma. Let U, V be affine varieties, $W \subset U$ any open set, then $W \times V$ in the above topology (“ T ”) is the topological space $W \times V$ with topology induced by that on $U \times V$.

Pf. Let $\Lambda \subset W \times V$ is T -open $\iff \forall$ affine $u' \subset W$ and affine $V' \subset V, \Lambda \cap (u' \times V') \subset u' \times V'$ is open.

But $\Lambda \subset U' \times V' \hookrightarrow U \times V' \hookrightarrow U \times V$, so Λ is T -open $\iff \forall$ open $E \subset U \times V$ the intersection $\Lambda \cap E \subset U \times V$ is open. We’re done, he thinks, I don’t know, wasn’t paying close attention.

Lemma. Let X, Y be prevarieties, $|Z|$ as before, T the above topology. Then \forall open affine $U \subset X, V \subset Y, U \times V \subset |Z|$ is T -open and the topology on $U \times V$ is that induced by T . [This will complete giving the topology on $X \times Y$.

Proof. (By picture) We know $U \times V \subset |Z|$ is T -open by the previous lemma. Now, look at $\Lambda \subset U \times V$ T -open. We know that $\Lambda \cap (U' \times V') \hookrightarrow U' \times V'$, but $\Lambda \hookrightarrow (U' \cap U) \times (V' \cap V) \hookrightarrow U' \times V'$ is open. This completes the lemma.

Cor. $|Z|$ with topology T is connected.

Pf. $|Z|$ is covered by $U \times V$'s which all intersect.

Lemma. (Sheaf O_Z). Let $K = k(X) \otimes_k k(Y)$. We know that K is an integral domain (by HW5 problem 2). For every $(x, y) \in |Z|$ let $O_{(x,y)}$ be the localization of $O_{X,x} \otimes O_{Y,y}$ at $m_x O_Y + m_y O_X$, which is a maximal ideal. Then, define $O_Z(U) = \bigcap_{(x,y) \in U} O_{(x,y)} \subset K$. If $U \subset X, V \subset Y$ are affine opens, then $O_Z|_{U \times V} = O_{U \times V}$.

This completes our proof that there are products in the category of prevarieties, and the way we did this, it's clear that the product of two affine varieties is again an affine variety under this product of prevarieties.

Remarks.

1. If $U \subset X$ is open, then $U \times Y \subset X \times Y$ is open. (Choose a covering and look at the intersections).
2. If $V \subset X$ is closed, then $V \times Y \subset X \times Y$ is closed. Note that $(V \times Y)^C = (V^C) \times Y$ which is open by the first remark.
3. If X, Y projective, then $X \times Y$ is also a projective variety. The second remark implies that it is enough to consider $X = \mathbb{P}^n, Y = \mathbb{P}^m$. The embedding goes $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{n+m+n}$ defined by $[x_0 : \dots : x_n] \times [y_0 : \dots : y_m] \mapsto [x_0 y_0 : \dots : x_n y_m]$. Call the coordinates u_{ij} . Look at $U_{x_i \neq 0} \times W_{y_j \neq 0} \rightarrow W_{u_{ij} \neq 0}$; this defines the map.

Example. Consider $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$. ($3 = 1 \cdot 1 + 1 + 1$). We get

$$k[u, v, s, t] \leftarrow k[x, y, z, w],$$

defined by $x \mapsto us, y \mapsto vt, z \mapsto ut, w \mapsto vs$. The kernel is $(xy - zw)$, a variety we've seen before!

1 Applications

So that's enough about the product existing. Why are they useful?

We had this bad example defined by gluing two copies of \mathbb{A}^1 by the natural isomorphism on $\mathbb{A}^1 - \{0\}$ lying in each one, which ends up being $Y = \mathbb{A} \cup \{*\}$ where $*$ lies at 0.

Consider two maps of prevarieties $f, g : X \rightarrow Y$. We'd expect that if f and g agree on points in X , then we want the set $f(X)$ to be closed, but in this example, this is not true! We have two maps into Y , one from inclusion in the first set, and one from inclusion in the other. These are the same on every point in $\mathbb{A}^1 - \{0\}$ but they are not the same on the "limit point" (0 or $*$).

Look at $\Delta : Y \rightarrow Y \times Y$ defined by lifting these two maps f and g . The map $\Delta : y \mapsto (y, y)$. Basically, we want this “diagonal” map Δ to be closed.

Note: The grader will be having office hours from now on on Tuesdays at 4pm in the Math Undergraduate lounge.