1 presheaves

Recall. A presheaf on a topological space X is a contravariant functor $F : Op(X) \to Set$ where $V \mapsto F(V)$.

Example: X, Y contravariant spaces, then if $F(V) = \{$cont. maps $U \to Y\}$ defines a presheaf.

Def. A presheaf F is a *sheaf* if for all collections $\{U_i\}$ of open sets the following sequence is exact:

$$F(\cup_i U_i) \xrightarrow{j} \Pi_i F(U_i) \rightrightarrows \Pi_{i,j} F(U_i \cap U_j).$$

Recall that a diagram of sets $S \xrightarrow{j} T \rightrightarrows R$ if j is injective and if for any $t \in T$ with $p_1(t) = p_2(t)$ there exists an $s \in S$ with $j(s) = t$.

This is rather confusing. What we mean is that the maps glue together.

That is, $f_i|_{U_i \cap U_j} = f_j|_{U_j \cap U_i}$.

If S, T, R were also group then $S \xrightarrow{j} T \rightrightarrows R$ exact means $0 \to S \xrightarrow{j} T \xrightarrow{p_1 - p_2} R$ is exact.

A restatement.

1. If $x_1, x_2 \in F(U)$ and $res_{UU_i} x_1 = res_{UU_i} x_2$ for all i, then $x_1 = x_2$, and
2. Given $x_i \in F(U_i)$ s.t. $res_{U_i U_i \cap U-j}(x_i) = res_{U_j U_i \cap U_j}(x_j) \forall i, j$ then there is an $x \in F(U)$ such that $x_i = res_U U_i(x) \forall i$.

Example. Let X, Y be topological spaces. Then $F(U) = \{$ continuous maps $U \to Y\}$ is a sheaf.

(1) is clear.

(2) We construct the map from X to Y by $x \mapsto f_i(x)$ for any i for which $x \in U_i$. This is unambiguous since these maps agree on the intersections, and we know this is continuous.

Ex. Let $f : X \to Y$ be continuous. Let $\mathcal{F}$ be a sheaf on X and define $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$. Or, to put it another way, we have

$$Op(Y) \xrightarrow{f^{-1}} Op(X) \xrightarrow{\mathcal{F}} Set,$$

so the composition $Op(Y) \xrightarrow{f_*\mathcal{F}} Set$ is the sheaf we're looking for. Let $V = \cup V_i, U = f^{-1}(V), U_i = f^{-1} V_i$.

Now, we want exactness for

$$f_*\mathcal{F}(V) \to \Pi_i f_*\mathcal{F}(V_i) \rightrightarrows \Pi_{i,j} f_*\mathcal{F}(V_i \cap V_j),$$

but these are just $\mathcal{F}(V) \to \pi_i \mathcal{F}(U_i) \rightrightarrows \pi_{i,j} \mathcal{F}(U_i \cap U_j)$, which is exact since $\mathcal{F}$ is a sheaf.

Ex. (presheaf but not a sheaf.) Suppose we have X, and $S \neq \{*\}$, a set, such that $F(U) = S$ for every U. We claim that $F(\emptyset)$ is a one-element set (axiom?) so this would only be a sheaf if $S \cong S \times S$, obviously not the case if S is finite but larger than 1 element.

Ex. Let X be a top. mspace, G a finite group with the discrete topology, and let $F(U) = \{$ cont. maps $U \to G\}/\{$constant maps $U \to G\}$. The point is that if X is not connected, then you might have a constant map on each connected component that is not a constant map globally, so $F(U) \to \Pi_i F(U_i)$ is not injective.

Theorem. Let X be a top. space, F a presheaf. Then $\exists$ a sheaf F^a with a map $F \to F^a$ which is universal for maps to sheaves. That is, if we have a map $F \to G$ of presheaves, there is a *unique* map $F^a \to G$ of sheaves. (From this, it automatically follows that F^a is unique up to isomorphism.

Def. If F is a presheaf, $x \in X$, then the *stalk* F_x at x is defined to be $\overset{\lim}{\to}_{x \in U} F(U)$. This $\overset{\lim}{\to}_{U \ni x} F(U)$ is the disjoint union over all $x \in U$ modulo that two elements are equivalent if they agree on restrictions to smaller and smaller neighborhoods of x.

Define $F^a(U) = \{(f_x)_{x \in U}, f_x \in F_x\}$, such that there is a U_i such that $\cup U_i = U$, and $f_i \in F(U_i)$ inducing $(f_x)_{x \in U_i}$. It is clear F^a is a sheaf: it was made to be a sheaf.

Ex. $X = \{a, b\}$ with the discrete topology, and the presheaf $F(U) = S$ for all U. Let $U_1 = \{a\}$ and $U_2 = \{b\}$. $F_a = F(U_1) = S$. Also, $F_b = F(U_2) = S$. $F^a(U_1) = S$. Similarly, $F^a(U_2) = S$. What is $F^a(U_1 \cup U_2)$? It must be $S \times S$ by our construction. This is no longer the constant presheaf, but is actually a sheaf.

Now, let us say we have a map $\alpha : F \to G$ of presheaves. We will now construct a map $F^a(U) \to G(U)$ that is a map of sheaves. We know how to map $(f_x)_{x \in U}$ into $\Pi_i G(U_i)$: we map it to $\alpha(f_i)$. We need to check that $\alpha(f_i) = \alpha(f_j)$ on $G(U_i \cap U_j)$.

Ex. X a locally connected top. space, $F(U) = S$. Then $F^a(U)$ is the disjoint union over connected components of U of S.

Notation Let F be a sheaf on X. We write $\Gamma(U, F)$ for $F(U)$, and call elements *sections of* F over U.

2 Back to Alg. Geom.

Let Σ be an affine alg. set, with the Zariski topology. Not all continuous maps $\Sigma_1 \to \Sigma_2$ are morphisms. What we need to do is consider not only the space but also a sheaf.

We will define a sheaf of rings (Σ, O_Σ).

Def. Let $X \subset k^n$ be an irreducible alg. set, $R = \Gamma(X)$. Let $K = \frac{(}{R})$. For each $x \in X$, let $m_x \subset R$ be the corresponding maximal ideal, and let $O_{X,x} = R_{m_x} = \{f/g | f, g \in R : g(x) \neq 0\}$. Define $O_X(U) = \cap_{x \in U} R_{m_x} \subset K$.