Fall 2010

Row Rank = Column Rank

This is in remorse for the mess I made at the end of class on Oct 1.

The *column rank* of an $m \times n$ matrix A is the dimension of the subspace of F^m spanned by the columns of A. Similarly, the *row rank* is the dimension of the subspace of the space F^n of *row vectors* spanned by the rows of A.

Theorem. *The row rank and the column rank of a matrix A are equal.*

proof. We have seen that there exist an invertible $m \times m$ matrix Q and an invertible $n \times n$ matrix P such that $A_1 = Q^{-1}AP$ has the block form

$$A_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

where I is an $r \times r$ identity matrix for some r, and the rest of the matrix is zero. For this matrix, it is obvious that *row rank* = *column rank* = r. The strategy is to reduce an arbitrary matrix to this form.

We can write $Q^{-1} = E_k \cdots E_2 E_1$ and $P = E'_1 E'_2 \cdots E'_\ell$ for some elementary $m \times m$ matrices E_i and $n \times n$ matrices E'_j. So A_1 is obtained from A by a sequence of row and column operations. (It doesn't matter whether one does the row operations before the column operations, or mixes them together: The associative law for matrix multiplication shows that $E(AE') = (EA)E'$, i.e., that row operations commute with column operations.)

This being so, it suffices to show that the row ranks and column ranks of a matrix A are equal to those of a matrix of the form EA, and also to those of a matrix of the form AE'. We'll treat the case of a row operation EA. The column operation AE' can be analyzed in a similar way, or one can use the transpose to change row operations to column operations.

We denote the matrix EA by A'. Let the columns of A be $C_1, ..., C_n$ and let those of A' be $C'_1, ..., C'_n$. Then $C'_j = EC_j$. Therefore any linear relation among the columns of A gives us a linear relation among the columns of A': If $C_1x_1 + \cdots + C_nx_n = 0$ then

$$E(C_1x_1 + \cdots + C_nx_n) = C'_1x_1 + \cdots + C'_nx_n = 0.$$

So if $j_1, ..., j_r$ are distinct indices between 1 and n, and if the set $\{C'_{j_1}, ..., C'_{j_r}\}$ is independent, the set $\{C_{j_1}, ..., C_{j_r}\}$ must also be independent. This shows that

$$\textit{column rank}(A') \leq \textit{column rank}(A).$$

Because the inverse of an elementary matrix is elementary and $A = E^{-1}A'$, we can also conclude that *column rank*$(A) \leq$ *column rank*(A'). The column ranks of the two matrices are equal.

Next, let the rows of A be $R_1, ..., R_n$ and let those of A' be $R'_1, ..., R'_n$, and let's suppose that E is an elementary matrix of the first type, that adds $a \cdot$ *row* k to *row* i. So $R'_j = R_j$ for $j \neq i$ and $R'_i = R_i + aR_k$. Then any linear combination of the rows R'_j is also a linear comination of the rows R_j. Therefore Span$\{R'_j\} \subset$ Span$\{R_j\}$, and so *row rank*$(A') \leq$ *row rank*(A). And because the inverse of E is elementary, we obtain the other inequality. Elementary matrices of the other types are treated easily, so the row ranks of the two matrices are equal. $\square$