18.440: Lecture 35

Martingales and the optional stopping theorem

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Outline

Martingales and stopping times

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Martingale definition

- Let S be the probability space. Let $X_0, X_1, X_2, \ldots$ be a sequence of real random variables. Interpret X_i as price of asset at ith time step.
- Say X_n sequence is a **martingale** if $E[|X_n|] < \infty$ for all n and $E[X_{n+1}|X_0, X_1, X_2, \ldots, X_n] = X_n$ for all n.
- "The expected price tomorrow is the price today."
- If you are given a mathematical description of a process $X_0, X_1, X_2, \ldots$ then how can you check whether it is a martingale?
- Consider all of the information that you know after having seen $X_0, X_1, \ldots, X_n$. Then try to figure out what additional (not yet known) randomness is involved in determining X_{n+1}. Use this to figure out the conditional expectation of X_{n+1}, and check to see whether this is always equal to the known X_n value.

Stopping time definition

- Let T be a non-negative integer valued random variable.
- Think of T as giving the time the asset will be sold if the price sequence is $X_0, X_1, X_2, \ldots$.
- Say that T is a **stopping time** if the event that $T = n$ depends only on the values X_i for $i \leq n$. In other words, the decision to sell at time n depends only on prices up to time n, not on (as yet unknown) future prices.

Martingale examples

- Suppose that $A_1, A_2, \ldots$ are i.i.d. random variables equal to -1 with probability .5 and 1 with probability .5.
- Let $X_0 = 0$ and $X_n = \sum_{i=1}^n A_i$ for $n > 0$. Is the X_n sequence a martingale?
- Answer: yes.
- What if each A_i is 1.01 with probability .5 and .99 with probability .5 and we write $X_0 = 1$ and $X_n = \prod_{i=1}^n A_i$ for $n > 0$? Then is X_n a martingale?
- Answer: yes.
- These are two classic martingale examples: a sum of independent random variables (each with mean zero) and a product of independent random variables (each with mean one).

Stopping time examples

- Let $A_1, \ldots$ be i.i.d. random variables equal to -1 with probability .5 and 1 with probability .5 and let $X_0 = 0$ and $X_n = \sum_{i=1}^n A_i$ for $n \geq 0$.
- Which of the following is a stopping time?
  1. The smallest T for which $|X_T| = 50$
  2. The smallest T for which $X_T \in \{-10, 100\}$
  3. The smallest T for which $X_T = 0$.
  4. The T at which the X_n sequence achieves the value 17 for the 9th time.
  5. The value of $T \in \{0, 1, 2, \ldots, 100\}$ for which X_T is largest.
  6. The largest $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T = 0$.
- Answer: first four, not last two.

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Optional stopping overview

- **Doob's optional stopping time theorem** is contained in many basic texts on probability and Martingales. (See, for example, Theorem 10.10 of *Probability with Martingales*, by David Williams, 1991.)
- Essentially says that you can't make money (in expectation) by buying and selling an asset whose price is a martingale.
- Precisely, if you buy the asset at some time and adopt any strategy at all for deciding when to sell it, then the expected price at the time you sell is the price you originally paid.
- If market price is a martingale, you cannot make money in expectation by "timing the market."

Doob's Optional Stopping Theorem: statement

- **Doob's Optional Stopping Theorem:** If the sequence $X_0, X_1, X_2, \ldots$ is a **bounded** martingale, and T is a stopping time, then the expected value of X_T is X_0.
- When we say martingale is bounded, we mean that for some C, we have that with probability one $|X_i| < C$ for all i.
- Why is this assumption necessary?
- Can we give a counterexample if boundedness is not assumed?
- Theorem can be proved by induction if *stopping time* T is bounded. Unbounded T requires a limit argument. (This is where boundedness of martingale is used.)

Martingales applied to finance

- Many asset prices are believed to behave approximately like martingales, at least in the short term.
- **Efficient market hypothesis**: new information is instantly absorbed into the stock value, so expected value of the stock tomorrow should be the value today. (If it were higher, statistical arbitrageurs would bid up today's price until this was not the case.)
- But what about interest, risk premium, etc.?
- According to the **fundamental theorem of asset pricing**, the discounted price $\frac{X(n)}{A(n)}$, where A is a risk-free asset, is a martingale with respected to **risk neutral probability**. More on this next lecture.

Martingales as successively revised best guesses

- The two-element sequence $E[X], X$ is a martingale.
- In previous lectures, we interpreted the conditional expectation $E[X|Y]$ as a random variable.
- Depends only on Y. Describes expectation of X given observed Y value.
- We showed $E[E[X|Y]] = E[X]$.
- This means that the three-element sequence $E[X], E[X|Y], X$ is a martingale.
- More generally if Y_i are any random variables, the sequence $E[X], E[X|Y_1], E[X|Y_1, Y_2], E[X|Y_1, Y_2, Y_3], \ldots$ is a martingale.

Martingales as real-time subjective probability updates

- Ivan sees email from girlfriend with subject "some possibly serious news", thinks there's a 20 percent chance she'll break up with him by email's end. Revises number after each line:
- Oh Ivan, I've missed you so much! 12
- I have something crazy to tell you, 24
- and so sorry to do this by email. (Where's your phone!?) 38
- I've been spending lots of time with a guy named Robert, 52
- a visiting database consultant on my project 34
- who seems very impressed by my work. 23
- Robert wants me to join his startup in Palo Alto. 38
- Exciting!!! Of course I said I'd have to talk to you first, 24
- because you are absolutely my top priority in my life, 8
- and you're stuck at MIT for at least three more years... 11
- but honestly, I'm just so confused on so many levels. 15
- Call me!!! I love you! Alice 0

More conditional probability martingale examples

- Example: let C be the amount of oil available for drilling under a particular piece of land. Suppose that ten geological tests are done that will ultimately determine the value of C. Let C_n be the **conditional expectation** of C *given* the outcome of the first n of these tests. Then the sequence $C_0, C_1, C_2, \ldots, C_{10} = C$ is a martingale.
- Let A_i be my best guess at the probability that a basketball team will win the game, given the outcome of the first i minutes of the game. Then (assuming some "rationality" of my personal probabilities) A_i is a martingale.

Are betting prices (as on Intrade) martingales?

- **Roughly yes, if markets efficient**, which should be in high-volume markets (with low bid-ask spread). Otherwise smart professionals could make money in expectation by buying/selling. But such efforts correct prices. (Note: overall "need for money" doesn't depend much outcome.)
- **Factors that might deter smart professionals**: betting fees, low trading volumes, risk of betting-site insolvency, government regulations impeding money flow. Also amount of money required to sit in deposit, whether it gets interest.
- **Question:** In low-volume market, might market manipulators (bidding up prices to make their candidates look better) overwhelm statistical arbitrageurs? If so, more money available for arbitrageurs who hang around.
- **Evidence for inefficiency:** price discrepancies, long shot bias.

Examples

- Suppose that an asset price is a martingale that starts at 50 and changes by increments of ± 1 at each time step. What is the probability that the price goes down to 40 before it goes up to 70?
- What is the probability that it goes down to 45 then up to 55 then down to 45 then up to 55 again — all before reaching either 0 or 100?

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