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1 Simple Poincaré Map for a limit cycle #02

Statement: Simple Poincaré Map for a limit cycle #02

Consider the following autonomous phase plane system

$$\left.\begin{aligned} \frac{dx}{dt} &= (x^2+y^4)\left(\nu\,x - \tfrac{\nu}{4}\,x^3 - x^2\,y - \nu\,x\,y^2 - 4\,y^3\right), \\ \frac{dy}{dt} &= (x^2+y^4)\left(\nu\,y + \tfrac{1}{4}\,x^3 - \tfrac{\nu}{4}\,x^2\,y + x\,y^2 - \nu\,y^3\right), \end{aligned}\right\} \quad \text{where } \nu > 0. \tag{1.1}$$

This system has a periodic solution **(show this)**, which can be written in the form

$$x = 2\cos\Phi, \quad y = \sin\Phi, \quad \text{where} \quad \frac{d\Phi}{dt} = 2\,(x^2+y^4) = 2\,(1+\cos^2\Phi)^2. \tag{1.2}$$

This solution produces an orbit going through the point $x = 0$, $y = 1$ in the phase plane. The orbit is an ellipse, as (1.2) shows.[1]

Construct (either numerically[2] or analytically) a Poincaré map near this orbit, and use it to show that the orbit is a stable limit cycle. *Define the Poincaré map* $\boldsymbol{z} \to \boldsymbol{u} = \boldsymbol{P}(\boldsymbol{z})$ as follows:

[1] Note that $\boldsymbol{\Phi}$ is a strictly increasing function of time.

[2] If you do it numerically, keep ν as a variable and check your answers for several values — say: $\nu = 0.1,\ 0.5,\ 1,\ 2,\ 5$.

- For every sufficiently small $\boldsymbol{z}$, let $\boldsymbol{x = X(t,\, z)}$ and $\boldsymbol{y = Y(t,\, z)}$ be the solution of (1.1) defined by $\boldsymbol{X(0,\, z) = 0}$ and $\boldsymbol{Y(0,\, z) = 1 + z}$.
- For this solution the polar angle $\boldsymbol{\theta}$ in the phase plane is an *increasing function of time*, starting at $\boldsymbol{\theta = \frac{1}{2}\,\pi}$ for $\boldsymbol{t = 0}$. Thus, there is a time $\boldsymbol{t = t_z}$ at which the solution reaches $\boldsymbol{\theta = \frac{5}{2}\,\pi}$ (note that $\boldsymbol{t_z}$ is a function of $\boldsymbol{z}$). Then take $\boldsymbol{u = Y(t_z,\, z) - 1}$.

Hint. Because t_z is a function of z, unknown a priori, the definition of the Poincaré map above is a bit awkward to implement. To avoid having to calculate t_z for each solution, it is a good idea to use a parameter other than time to describe the orbits. For example, if the equations are written in terms of a parameter such as the polar angle — namely $\boldsymbol{\frac{dx}{d\theta} = F(x,\, y)}$ and $\boldsymbol{\frac{dy}{d\theta} = G(x,\, y)}$, then the Poincaré map is easier to describe, as θ varies from $\theta = \frac{1}{2}\,\pi$ to $\theta = \frac{5}{2}\,\pi$ in every one of the orbits needed to compute $u = P(z)$. Note that this is just a "for example", using the polar angle is not the best choice. *You may want to scale the variables first, so that the limit circle is a circle, not an ellipse.* ♣

2 Bifurcations in the torus #01

Statement: Bifurcations in the torus #01

Bifurcations in the torus, phase-locking, and oscillator death. This problem is based on a paper on systems of neural oscillators by G. B. Ermentrout and N. Kopell,[3] where they illustrate the notion of *oscillator death* (see § 3) with the following model

$$\dot{\theta}_1 = \omega_1 + \sin\theta_1\,\cos\theta_2 \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + \sin\theta_2\,\cos\theta_1, \tag{2.1}$$

where $\boldsymbol{\omega_1,\, \omega_2 > 0}$. *Here θ_1 and θ_2 are to be interpreted as the phases of two coupled stable and attracting limit cycle oscillators, which are assumed to "survive" the coupling, so that the notion of their "individual phases" remains* — see § 3.

a. Classify all the different behaviors that the solutions to (2.1) have, as the parameters vary in the positive quadrant of the $[\omega_1,\, \omega_2]$–plane. **Do a diagram** in this quadrant, indicating the regions that correspond to each behavior.

   The final answer should look something like this: *(i) In such and such region the solutions are attracted to a limit cycle [Note that this is phase locking]. (ii) In such and such region the solutions are attracted to a stable node [Note that this is oscillator death]. (iii) In such and such region the solutions are quasi-periodic with two periods [Phase locking fails]. (iv) ...*
   Plus a drawing of the regions ... will all the statements properly justified.

b. Draw the bifurcation curves in the $[\omega_1,\, \omega_2]$–plane. **Describe** each bifurcation.

[3] *Oscillator death in systems of coupled neural oscillators.* SIAM J. Appl. Math. **50**:125 (1990).

Hints. *I did not find an elegant way to analyze the system geometrically. The hints below lead you to an approach that is (mostly) analytical, but allows a systematic and thorough investigation.*

h1. Consider the equations satisfied by $\phi = \theta_1 + \theta_2$ and $\psi = \theta_1 - \theta_2$.

h2. You may find the following result useful

Let $\boldsymbol{\alpha > 1}$. Then the solutions to the equation $\boldsymbol{\dot{\chi} = \alpha + \sin\chi}$ can be written in the form $\boldsymbol{\chi = \mu\,(t - t_0) + X(\mu\,(t - t_0))}$, where $\boldsymbol{\mu > 0}$ is a constant, $\boldsymbol{X}$ is $2\,\pi$-periodic, $\boldsymbol{X(0) = 0}$, and t_0 is an arbitrary constant.

Furthermore: $\boldsymbol{\mu}$ is an increasing function of $\boldsymbol{\alpha}$, with $\boldsymbol{\lim_{\alpha\to 1}\mu = 0}$ and $\boldsymbol{\lim_{\alpha\to\infty}\mu = \infty}$.

All this follows from the results in § 2.1, upon using a change of variables that transforms $\dot{\chi} = \alpha + \sin\chi$ into (2.2). In particular, note that the "μ" in § 2.1 (call it $\tilde{\mu}$) is related to the one here by $\mu = \alpha\,\tilde{\mu}$, with $\kappa = 1/\alpha$.

2.1 Notes on first order equation with a periodic right hand side

These are notes with facts useful for this problem. They are not a problem.

Consider the equation

$$\boldsymbol{\dot{\phi} = 1 - \kappa\,\sin\phi,} \quad \text{where } \boldsymbol{0 < \kappa < 1.} \tag{2.2}$$

Since $\dot{\phi} \geq 1 - \kappa > 0$, ϕ *is monotone increasing.* **The statements below apply.**

1. There is a constant $0 < \mu < 1$, and a function $\Phi = \Phi(\zeta)$ — periodic of period $2\,\pi$ — such that any solution to (2.2) has the form

$$\boldsymbol{\phi = \mu\,(t - t_0) + \Phi(\mu\,(t - t_0)),} \tag{2.3}$$

where t_0 is a constant and $\boldsymbol{\Phi(0) = 0}$.

2. Note that $\sin(\phi)$ is periodic in t, of period $\boldsymbol{T = \frac{2\,\pi}{\mu}}$, with

$$\boldsymbol{M =} \textbf{ average}\boldsymbol{(\sin\phi) = \frac{1-\mu}{\kappa} > 0,} \tag{2.4}$$

where M is defined by the first equality — $M = M(\kappa)$ only, since μ depends on κ only.

3. Let $\boldsymbol{\phi_*}$ **be the solution to (2.2) defined by** $\boldsymbol{\phi_*(0) = 0}$ — i.e.: set $t_0 = 0$ in (2.3). Then

$$\Theta(\mu\,t) = \int_0^t (\sin(\phi_*(s)) - M)\,ds = -\frac{1}{\kappa}\,\Phi(\mu\,t), \tag{2.5}$$

where Θ is defined by the first equality.

4. Assume that $0 < \kappa \ll 1$. Then a Poincaré-Lindstedt expansion yields

$$\phi_* = \mu\,t - \kappa\,(1 - \cos(\mu\,t)) + O(\kappa^2) \quad \text{and} \quad \mu = 1 - \frac{1}{2}\,\kappa^2 + O(\kappa^4). \tag{2.6}$$

It follows that $T = 2\,\pi + \pi\,\kappa^2 + O(\kappa^4)$ and $M = \dfrac{1}{2}\,\kappa + O(\kappa^3)$.

5. Assume that $0 < 1 - \kappa \ll 1$. Then
$$\boldsymbol{\mu = O(\sqrt{1-\kappa})} \tag{2.7}$$

In case you are curious as to how to show the above applies, here are some hints.

a. Define $T > 0$ as the (unique) time at which $\phi_*(T) = 2\,\pi$ — why is the solution unique?

b. Show that $\phi_*(t+T) = 2\,\pi + \phi_*(t)$ — sub-hint: both sides are solutions!

c. Define Φ by $\Phi(\mu\,t) = \phi_*(t) - \mu\,t$, with $\mu = 2\,\pi/T$, and show that Φ is periodic of period $2\,\pi$.

d. Write the general solution in terms of ϕ_*.

e. Show that $T = O\big(1/\sqrt{1-\kappa}\big)$ as $\kappa \to 1$ — sub-hint: critical slowing-down.

f. To show that $\mu < 1$, use (2.2) and separation of variables to write T as an integral over ϕ from 0 to $2\,\pi$. Then show $T > 2\,\pi$

g. To show (2.4), take the average of (2.2).

h. To obtain the second equality in (2.5), substitute $\phi_* = \mu\,t + \Phi(\mu\,t)$ into (2.2), and obtain a formula for $\sin(\phi_*)$ in terms of Φ.

3 Notes: coupled oscillators, phase locking, etc.

These are notes with facts useful for the problems. They are not a problem.

3.1 On phases and frequencies

Consider a system made by two coupled oscillators, where each of the oscillators (when not coupled) has a stable attracting limit cycle. Let the limit cycle solutions for the two oscillators be given by $\boldsymbol{\vec{x}_1 = \vec{F}_1(\omega_1\,t)}$ and $\boldsymbol{\vec{x}_2 = \vec{F}_2(\omega_2\,t)}$, where $\boldsymbol{\vec{x}_1}$ and $\boldsymbol{\vec{x}_2}$ are the vectors of variables for each of the two systems, the $\boldsymbol{\vec{F}_j}$ *are periodic functions of period* $2\,\pi$, and the ω_j are constants (related to the limit cycle periods by $\omega_j = 2\,\pi/T_j$). In the un-coupled system, the two limit cycle orbits make up a *stable attracting invariant torus* for the evolution. Assume now that either the coupling is weak, or that the two limit cycles are strongly stable. *Then the stable attracting invariant torus survives for the coupled system.*[4] The solutions (on this torus) can be (approximately) represented by

$$\vec{x}_1 \approx \vec{F}_1(\theta_1) \quad \text{and} \quad \vec{x}_2 \approx \vec{F}_2(\theta_2), \tag{3.1}$$

where $\boldsymbol{\theta_1 = \theta_1(t)}$ and $\boldsymbol{\theta_2 = \theta_2(t)}$ satisfy some equations, of the general form

$$\dot{\theta}_1 = \omega_1 + K_1(\theta_1,\, \theta_2) \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + K_2(\theta_1,\, \theta_2). \tag{3.2}$$

[4] With a (slightly) changed shape and position.

Here K_1 and K_2 are the "projections" of the coupling terms along the oscillator limit cycles. For example, take $K_1(\theta_1, \theta_2) = \sin\theta_1 \cos\theta_2$ and $K_2(\theta_1, \theta_2) = \sin\theta_2 \cos\theta_1$. Another example is the one in § 8.6 of Strogatz' book (*Nonlinear Dynamics and Chaos*), where a model system with

$$K_1(\theta_1, \theta_2) = -\kappa_1 \sin(\theta_1 - \theta_2) \text{ and } K_2(\theta_1, \theta_2) = \kappa_2 \sin(\theta_1 - \theta_2)$$

is introduced, with constants $\kappa_1, \kappa_2 > 0$. Note that:

1. In (3.2), K_1 and K_2 must be 2π-periodic functions of θ_1 and θ_2.
2. The **phase space for (3.2) is the invariant torus $\mathcal{T}$**, on which θ_1 and θ_2 are the angles. We can also think of $\mathcal{T}$ as a $2\pi \times 2\pi$ square with its opposite sides identified. On $\mathcal{T}$ a **solution is periodic** if and only if $\theta_1(t+T) = \theta_1(t) + 2n\pi$ and $\theta_2(t+T) = \theta_2(t) + 2m\pi$, where $T > 0$ is the period, and both n and m are integers.
3. In the "Coupled oscillators # 01" problem an example of the process leading to (3.2) is presented.
4. The $\boldsymbol{\theta_j}$'s are the **oscillator phases.** One can also define oscillator frequencies, even when the θ_j's do not have the form $\theta_j = \omega_j t$, with ω_j constant. The idea is that, near any time t_0 we can write $\theta_j = \theta_j(t_0) + \dot{\theta}_j(t_0)(t - t_0) + \ldots$, identifying $\dot{\theta}_j(t_0)$ as the local frequency. Hence, we define the **oscillator frequencies by $\boldsymbol{\tilde{\omega}_j = \dot{\theta}_j}$**. These frequencies are, of course, **generally not constants.**
5. The notion of phases can survive <u>even if</u> the limit cycles cease to exist (i.e.: oscillator death). For example: if the equations for θ_1 and θ_2 have an attracting critical point. We will see examples where this happens in the problems, e.g.: "Bifurcations in the torus # 01".

3.2 Phase locking and oscillator death

The coupling of two oscillators, each with a stable attracting limit cycle, can produce many behaviors. Two of particular interest are

1. Often, if the frequencies are close enough, the system **phase locks.** This means that a stable periodic solution arises, in which both oscillators run at some composite frequency, with their phase difference kept constant. The composite frequency need not be constant. In fact, it may periodically oscillate about a constant average value.
2. However, the coupling may also suppress the oscillations, with the resulting system having a stable steady state. This even if none of the component oscillators has a stable steady state. This is **oscillator death.** It can happen not only for coupled pairs of oscillators, but also for chains of oscillators with coupling to the nearest neighbors.

On the other hand, we note that it is also possible to produce an oscillating system, with a stable oscillation, by coupling non-oscillating systems (e.g., the coupling of excitable systems can do this).

THE END.