

Problem Set Number 4, 18.385j/2.036j

MIT (Fall 2014)

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1 Problem 06.08.09 - Strogatz (Counter-rotating limit cycles)

Statement for problem 06.08.09

A smooth vector field on the phase plane is known to have exactly two closed trajectories, one of which lies inside the other. The inner circle runs counterclockwise, and the outer runs clockwise.

True or False: *there must be at least one fixed point in the region between the cycles.*

If true, prove it. If false, provide a simple counterexample.

Hint: Beware of “gut feeling” instinctive answers. There is a good chance that your intuition is wrong!

2 Index theory - interpolating from saddles to nodes #2

Statement: Index theory - interpolating from saddles to nodes #2

Consider a one parameter family of phase plane systems

$$\dot{x} = f(x, y, r) \quad \text{and} \quad \dot{y} = g(x, y, r), \tag{2.1}$$

where f and g are smooth functions of all of its arguments — including the parameter r . Assume that:

1. The origin is a critical point for all values of r . That is $f(0, 0, r) = g(0, 0, r) = 0$.
2. For $r = 0$ the origin is an isolated critical point. In fact, a saddle.
3. For $r = 1$ the origin is an isolated critical point. In fact, a node.

Show that there is at least one value $0 < R < 1$, such that: for $r = R$ the origin is not isolated critical point.

Hint. Let $\mathcal{I} = \mathcal{I}(r)$ be the index of the critical point at the origin for (2.1), for any r for which it is defined. Also note that $\mathcal{I}(0) = -1$ and $\mathcal{I}(1) = 1$. Use now the properties of the index.

3 A single critical point between two limit cycles #1

Statement: A single critical point between two limit cycles #1

Consider a phase plane system

$$\dot{x} = f(x, y) \quad \text{and} \quad \dot{y} = g(x, y), \quad (3.2)$$

where f and g are smooth functions of all of its arguments. Assume that:

1. The system has **isolated critical points only**.
2. The system has **exactly two limit cycles**, Γ_1 and Γ_2 , with Γ_2 **enclosing** Γ_1 .
3. Between Γ_1 and Γ_2 there is a **single critical point** \mathcal{C} .

Given the above:

- A. Calculate the index \mathcal{I} for \mathcal{C} .**
- B. If A is the matrix of the linearized system near \mathcal{C} , show that $\Delta = \det(A) = 0$.**
- C. Construct an example of this situation, and sketch its phase plane portrait.** Hint h2 below leads you to the construction of a system of this form that will have a cycle graph (produced by an homoclinic orbit connecting \mathcal{C} to itself). **Sketch a phase plane portrait of the neighborhood of \mathcal{C} . How would you “classify” this critical point?**

Hints:

- h1. The index of a critical point is 1 if $\Delta > 0$. This follows because then the critical point is either a node, a spiral point, satisfies¹ $\Delta = \tau^2/4$ (linearly degenerate node), or has $\tau = 0$ (linearized center). In the first two cases the answer is obvious. For the other two cases an arbitrarily small perturbation can be used to transform the critical point into a spiral point, and then the continuity of the index yields the desired result.*

¹ Here τ is the trace of the linearization matrix at the critical point, while Δ is the determinant.

h2. Consider systems of the form $\dot{x} = a x - b y$ and $\dot{y} = b x + a y$, which are equivalent to $\dot{r} = r a$ and $\dot{\theta} = b$, where r and θ are the polar coordinates (**show this!**) To do part C select $a = G(x^2 + y^2)$ and $b = b(x, y)$ appropriately. For example: take b such that $\dot{\theta} = b > 0$ everywhere, except for a single point C between the circles $r = 1$ and $r = 3$. Then choose a so that limit cycles occur at $r = 1$ and $r = 3$, and so that C is a critical point. The reason for selecting $a = G(r^2)$ as a function of r^2 , and not r , is to avoid possible singular behavior at $r = 0$, where $r = r(x, y) = \sqrt{x^2 + y^2}$ is not differentiable.

4 Problem 06.01.09 - Strogatz (Computer generated phase portrait)

Statement for problem 06.01.09

Plot a computer generated phase plane portrait for the “Dipole fixed point” system

$$\frac{dx}{dt} = 2xy \quad \text{and} \quad \frac{dy}{dt} = y^2 - x^2. \quad (4.1)$$

THE END.

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