Chapter 6

Computerized tomography

6.1 Assumptions and vocabulary

(...)

Computerized tomography (CT scans, as well as PET scans) imaging involves inversion of a Radon or X-ray transform. It is primarily used for medical imaging.

In two spatial dimensions, the variables in the Radon domain are t (offset) and θ (angle). Data in the form $d(t, \theta)$ corresponds to the *parallel beam* geometry. More often, data follow the *fan-beam* geometry, where for a given value of θ the rays intersect at a point (the source of X-rays), and t indexes rays within the fan. The transformation to go from parallel-beam to fan-beam and back is

$$d_{\text{fan}}(t, \theta) = d_{\text{para}}(t, \theta + (at + b)),$$

for some numbers a and b that depend on the acquisition geometry. Datasets in the Radon domain are in practice called *sinograms*, because the Radon transform of a Dirac mass is a sine wave[1].

6.2 The Radon transform and its inverse

Radon transform:

$$(Rf)(t, \theta) = \int \delta(t - x \cdot e_\theta) f(x) \, dx,$$

[1] More precisely, a distribution supported on the graph of a sine wave, see an exercise at the end of the chapter.

with $e_\theta = (\cos\theta, \sin\theta)^T$.

Fourier transform in t / Fourier-slice theorem[2]:

$$\widehat{Rf}(\omega, \theta) = \int e^{-i\omega x \cdot e_\theta} f(x)\, dx.$$

Adjoint Radon transform / (unfiltered) backprojection:

$$\begin{aligned} R^* d(x) &= \int e^{i\omega x \cdot e_\theta} \widehat{d}(\omega, \theta)\, d\omega d\theta \\ &= \int \delta(t - x \cdot e_\theta) d(t, \theta)\, dt d\theta \\ &= \int d(x \cdot \theta, \theta)\, d\theta \end{aligned}$$

Inverse Radon transform / filtered backprojection in the case of two spatial dimensions:

$$R^{-1} d(x) = \frac{1}{(2\pi)^n} \int e^{i\omega x \cdot e_\theta} \widehat{d}(\omega, \theta)\, \omega\, d\omega d\theta.$$

(notice the factor ω.)

Filtered backprojection can be computed by the following sequence of steps:

- Take a Fourier transform to pass from t to ω;
- Multiply by ω;
- Take an inverse Fourier transform from ω back to t, call $D(t, \theta)$ the result;
- Compute $\int d(x \cdot \theta, \theta)\, d\theta$ by quadrature and interpolation (piecewise linear interpolation is often accurate enough.)

[2]The direct Fourier transform comes with $e^{-i\omega t}$. Here t is offset, not time, so we use the usual convention for the FT.

6.3 Exercises

1. Compute the Radon transform of a Dirac mass, and show that it is nonzero along a sinusoidal curve (with independent variable θ and dependent variable t, and wavelength 2π.)

2. In this problem set we will form an image from a fan-beam CT dataset. (Courtesy Frank Natterer)

   Download the data set at http://math.mit.edu/icg/ct.mat

   and load it in MATLAB® with `load ct.mat`

   The array g is a sinogram. It has 513 rows, corresponding to uniformly sampled offsets t, and 360 columns, corresponding to uniform, all-around angular sampling with 1-degree steps in θ. The acquisition is fan-beam: a transformation is needed to recover the parallel-beam geometry. The fan-beam geometry manifests itself in that the angle depends on the offset t in a linear fashion. Instead of being just θ, it is ($1 \leq t \leq 513$ is the row index)

   $$\theta + \frac{t - 257}{256}\alpha,$$

   with

   $$\sin \alpha = \frac{1}{2.87}.$$

   Imaging from a parallel-beam sinogram is done by filtered backprojection. Filtering is multiplication by ω in the ω domain dual to the offset t. Backprojection of a sinogram $g(t, \theta)$ is

   $$I(x) = \sum_{\theta} g(x \cdot \mathbf{e}_\theta, \theta),$$

   where $\mathbf{e}_\theta$ is $(\cos\theta, \sin\theta)^T$. Form the image on a grid which has at least 100 by 100 grid points (preferably 200 by 200). You will need an interpolation routine since $x \cdot \mathbf{e}_\theta$ may not be an integer; piecewise linear interpolation is accurate enough (`interp1` in MATLAB).

   In your writeup, show your best image, your code, and write no more than one page to explain your choices.