Two recent crossing number results

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1 Introduction: Why this

I decided to look into different crossing numbers and recent research into the relations between them. Many advances in crossing numbers seem to be in terms of adjusting constants, rather than finding new relations. I found many papers, for example, on tightening the bound on the crossing number of K_n or $K_{m,n}$. Others, more interesting to me, worked on what has been described as the "theory of large graphs" (Székeley). Two recent results, involving variations on the pair crossing and odd crossing numbers, are what I will focus on here. While the two papers seem to have little in common, both revisit Tutte's original models to find new ways of evaluating or bounding crossing numbers. Perhaps there is more to discover by going back to the problem's forgotten roots.

2 Lots and lots of crossing numbers: A review

Although crossing numbers have been studied for over fifty years, only within the last ten years or so have they been carefully defined. Pach and Tóth, in their paper, defined four distinct characteristics of a graph, each of which could be called crossing number. Tutte worked with yet another type of crossing number, and a new result involves a sixth kind.

The crossing number is defined as a minimum over all drawings, but what drawings are allowable? It's assumed that the drawings considered have vertices at distinct points. Edges are Jordan curves. All crossings are proper (i.e. edges intersect at a finite number of points, and are never tangent to each other). Each crossing involves only two edges (i.e. every non-vertex point is in at most two edges).

The crossing number, $cr(G)$, of a graph G is the minimum (over all allowable drawings) of the number of crossings. The rectilinear crossing number, $rcr(G)$, is the minimum number of crossings over all allowable drawings with straight lines as edges. The odd crossing number, $oddcr(G)$, is the minimum number of pairs of edges which cross an odd number of times over all drawings. The paircrossing number, $prcr(G)$, is the minimum number of pairs of edges which cross. Clearly

$$oddcr(G) \leq prcr(G) \leq cr(G) \leq rcr(G)$$

Another recent paper looked at Tutte's initial, unproved assumption that in a odd-crossing-minimizing drawing of G, no adjacent edges cross. This paper looks at $ioddcr(G)$, the independent odd crossing number of G. $ioddcr(G)$ is defined to be the minimum over all drawings of G of the pairs of non-adjacent edges which cross an odd number of times. That is, crossings are only counted if they occur an odd number of times and involve non-adjacent edges.

Yet another crossing number is Tutte's algebraic crossing number, $acr(G)$. To calculate $acr(G)$, first fix some orientation of $E(G)$. For each non-adjacent pair of edges e_1, e_2 in G, let $\lambda(e_1, e_2)$ be the difference between the number of left-to-right crossings of e_1 on e_2 and right-to-left crossings of e_1 on e_2. Then the algebraic crossing number of a graph is defined by

$$acr(G) = \min_D \sum_{\{e_1,e_2\} \in \binom{E}{2}} \mid \lambda(e_1, e_2) \mid$$

where the minimum is taken over all drawings D of the graph G. Since e_1 and e_2 crossing an odd number of times implies $\mid \lambda(e_1, e_2) \mid$ is at least one, we know

$$ioddcr(G) \leq acr(G)$$

Also, $\mid \lambda(e_1, e_2) \mid$ is at most the total number of times e_1 and e_2 cross, so we have

$$acr(G) \leq cr(G)$$

It was conjectured by Pach and Tóth (and others) that $oddcr(G) = prcr(G) = cr(G) = rcr(G)$. This question remained open for years, until a recent paper (using $acr(G)$) showed that in fact

$$oddcr(G) \neq cr(G)$$

3 $oddcr(G) \neq cr(G)$: What a surprise!

It seems reasonable to suppose $oddcr(G) = cr(G)$. It's true that, given a drawing of G that demonstrates $oddcr(G) = 0$, it's possible to construct a drawing of G that demonstrates $cr(G) = 0$. It was thought this construction could be expanded to a construction converting an odd-crossing-minimizing drawing to a crossing-minimizing drawing. However, that turns out not to be the case. In fact, there exists an infinite family of graphs for which the crossing number and the odd crossing number are not even asymptotically equal.

As in the other one summarized here, this proof proceeds by transforming a graph drawing into a different object, proving a bound on that object, and then changing back to graph drawings. In this case, the new object is a weighted map.

A weighted map M is a 2-manifold S and a set $P = \{a_1, \ldots a_m, b_1, \ldots b_m\}$ of points on ∂S with positive weights $w_1 \ldots w_m$. A realization R of this map is a set of curves $\{\gamma_1, \ldots \gamma_m\}$ such that γ_i connects a_i to b_i. Let $i(\gamma_i, \gamma_j)$ be the geometric intersection number of the two curves and $[x]$ be 1 if x is true, 0 otherwise. Then we can define crossing numbers for a realization R as follows:

$$cr(R) = \sum_{1 \le k \le l \le m} i(\gamma_k, \gamma_l) w_k w_l$$

$$prcr(R) = \sum_{1 \le k \le l \le m} [i(\gamma_k, \gamma_l) > 0] w_k w_l$$

$$oddcr(R) = \sum_{1 \le k \le l \le m} [i(\gamma_k, \gamma_l) \equiv_2 1] w_k w_l$$

The corresponding crossing numbers for a weighted map M are found by minimizing each equation over all possible realizations R.

(Note that it's possible to transform a realization of a map into a graph drawing. For example, contract the boundary of S, let the vertices be (equivalence classes of) points in P, and (multi-)edges be the curves γ_i. Similarly, a graph drawing can be transformed into a map realization. Remove a disc of radius ϵ around each vertex, let $\{\gamma_i\}$ be the set of edges, and let P be the edges' intersections with the disc boundaries.)

To prove the inequality of *oddcr* and *cr*, we'll find a family of weighted maps whose crossing numbers are not equal to their odd crossing numbers. Then we transform this into a family of weighted graphs whose crossing-minimizing drawings are the same as the minimizing realizations. Finally, we'll transform the weighted graphs to ordinary graphs, while still preserving the inequality.

In practice, we only actually care about realizations in which S is the annulus, $\{a_i\}$ is on the inner ring, and $\{b_i\}$ is on the outer ring. These are the realizations that will eventually be transformed into graph drawings. In a crossing-minimizing realization, a curve γ_i can be entirely described by its twist number k_i, where integer k_i represents the number of clockwise (positive) or counterclockwise (negative) times γ_i goes around the annulus. Given only $a_i, b_i, a_j, b_j, k_i, k_j$, it's possible to calculate $i(\gamma_i, \gamma_j)$, and hence the minimum number of crossings.

Let $D^k(\gamma_i)$ be the curve produced by changing γ_i's twist by k. That is, let $D^k(\gamma_i)$ be the curve with the same endpoints as γ_i and twist number $k_i + k$. Orient all the curves γ_i from a_i to b_i. Then

$$\lambda(D^k(\gamma_i), D^l(\gamma_j)) = k - l + \lambda(\gamma_i, \gamma_j)$$

Furthermore, for crossing-minimizing maps of this form,

$$i(\gamma_i, \gamma_j) = \mid \lambda(\gamma_i, \gamma_j) \mid$$

This can be used to develop formulae for calculating the crossing numbers of maps M on the annulus. Fix $\{a_1 \dots a_n\}$ and $\{b'_1 \dots b'_n\}$ on the boundary of an annulus, as described, in clockwise order. Fix some curve γ_0 from the inner to the outer boundary with endpoints between a_n and a_1 and b_n and b_1. For a fixed permutation $\sigma \in S_n$, let $b_i = b'_{\sigma(i)}$. This gives us M_σ. Now let γ'_i be a curve connecting a_i and b_i such that $i(\gamma_0, \gamma'_i) = 0$. Then a sequence of integers $\bar{x} = x_1, \dots x_n$ determines a realization $R_{\bar{x}}$ in which $\gamma_i = D^{x_i}(\gamma'_i)$. Similarly, every realization R can be encoded by a fixed permutation σ_R, and a sequence $\bar{x}_R$.

Recall the discussion of $acr(G)$, above. Note that the permutation determines the algebraic crossing number. For $i < j$, $\lambda(\gamma'_i, \gamma'_j) = [\sigma(i) > \sigma(j)]$. Therefore

$$i(\gamma_i, \gamma_j) = x_i - x_j + [\sigma(i) > \sigma(j)]$$

So instead of minimizing over all realizations R of M_σ, we can instead minimize over all vectors $\bar{x}$ to calculate the crossing numbers. This gives us

$$acr(M_\sigma) = cr(M_\sigma) = \min_{\bar{x}} \sum_{i<j} \mid x_i - x_j + [\sigma(i) > \sigma(j)] \mid w_i w_j$$

$$prcr(M_\sigma) = \min_{\bar{x}} \sum_{i<j} [(x_i - x_j + [\sigma(i) > \sigma(j)]) \neq 0] w_i w_j$$

$$oddcr(M_\sigma) = \min_{\bar{x}} \sum_{i<j} [(x_i - x_j + [\sigma(i) > \sigma(j)]) \equiv_2 1] w_i w_j$$

A slackening of these equations, in which $\bar{x}$ is allowed to contain non-integers, can be minimized in polynomial time by running the corresponding linear programs. Furthermore, the minimum is achievable by integral $\bar{x}$.

The family of weighted maps corresponding to graphs with unequal crossing numbers is generated by one simple example. Consider the weighted map M with $S = \{re^{i\theta} \mid 1 \le r \le 2 \text{ and } \theta \in [0, 2\pi]\}$, $a_j = i^j$ $(j = 1,2,3,4)$, $b_1 = 2i$, $b_2 = 2$, $b_3 = -2i$, $b_4 = -2$, and weights $\{w_i\}$. Then if $w_1 \le w_2 \le w_4 \le w_3$ and $w_1 + w_4 \ge w_3$, we have $cr(M) = prcr(M) = w_1 w_4 + w_2 w_3$ and $oddcr(M) = w_2 w_4 + w_1 w_3$ (by minimizing the linear programs, as above). So if $w_1 w_4 + w_2 w_3 \ne w_2 w_4 + w_1 w_3$, then $oddcr(M) \ne cr(M)$. Optimizing the gap over integer weights gives us $\{M_m\}$ with weights $w_2 = w_4 = m$, $w_1 = \lfloor (\frac{\sqrt{3}-1}{2}) m \rfloor$, and $w_3 = \lfloor (\frac{\sqrt{3}+1}{2}) m \rfloor$.

We will create our family of graphs $\{G_m\}$ with unequal crossing numbers out of $\{M_m\}$. First, create a weighted map N by replacing each a_i (resp. b_i) with a sequence $(a_{i,1}, \ldots a_{i,w_i})$ (resp. $(b_{i,1} \ldots b_{i,w_i})$) of points going clockwise along an arc of length ϵ along ∂S centered at a_i (resp. b_i) and setting $w_{i,j} = 1$ $\forall (i,j)$. In a crossing-minimizing realization of N, the newly created clusters of edges run in parallel, so $cr(N) = cr(M)$ and $oddcr(N) = oddcr(M)$.

Now transform N into a graph G. Let

$$V(G) = \{a_{i,j}\} \cup \{b_{i,j}\}$$

Let $e_{i,j,k}$ (resp. $f_{i,j,k}$) be the k^{th} copy of an edge between $a_{i,j}$ (resp. $b_{i,j}$), and $a_{i,j+1}$ (resp. $b_{i,j+1}$) for $j < w_i$, and the k^{th} copy of an edge between $a_{i,j}$ (resp. $b_{i,j}$) and $a_{i+1,1}$ (resp. $b_{i+1,1}$) for $j = w_i$. Then let

$$\begin{aligned} E(G) = &\{\{a_{i,j}, b_{i,j}\} \mid (i,j) \in [4] \times [w_i]\} \\ &\cup \{e_{i,j,k} \mid (i,j,k) \in [4] \times [w_i] \times [cr(M)+1]\} \\ &\cup \{f_{i,j,k} \mid (i,j,k) \in [4] \times [w_i] \times [cr(M)+1]\} \end{aligned}$$

That is, we create G by letting the arcs around the boundary of the anulus have weight $cr(M)+1$, letting P be the vertices, and replacing curves of weight w with w edges between two vertices. Because of how the graph is constructed, in a crossing minimizing drawing the edges between $a_{i,j}$ and $b_{i,j}$ run parallel to each other, and the edges within $\{a_{i,j}\}$ and within $\{b_{i,j}\}$ must not cross each other (and so can be deformed into the inner and outer rings of ∂S without changing the crossings). Therefore a minimizing drawing of G is very similar to a minimizing drawing of N, and hence a minimizing drawing of M. However, it's possible for G to have a minimizing drawing with orientation reversed. That is, $\{a_{i,j}\}$ could be in clockwise order and $\{b_{i.j}\}$ counterclockwise. Let N' be the analogous weighted map, i.e. N with the order of $\{a_{i,j}\}$ reversed. In any realization of N', because of the order reversal, $\{\{a_{i,j}, b_{i,j}\} \mid j \in [w_i]\}$ must have many odd crossings $\forall i$. In fact, $ocr(N') \geq 2m^2 - 4m$.

Since any realization of N generates a drawing of G with the same crossings, and we know from above that $ocr(N) = w_1w_3 + w_2w_4$, we have

$$ocr(G) \leq ocr(N) \leq w_1w_3 + w_2w_4 \leq \frac{3}{2}m^2$$

Similarly, any drawing of G generates a realization of either N or N', so we have

$$cr(G) \geq \min\{cr(N), cr(N')\}$$

Since $ocr(N') \leq cr(N)$ and we have the above bounds on $cr(N)$ and $ocr(N')$, we get (for m sufficiently large)

$$\begin{aligned} cr(G) &\geq \min\{cr(N), cr(N')\} \\ &\geq \min\{w_1w_4 + w_2w_3, ocr(N')\} \\ &\geq \min\{\sqrt{3}m^2 - 2m, 2m^2 - 4m\} \\ &\geq \sqrt{3}m^2 - 2m \end{aligned}$$

This gives us a whole family of graphs $\{G_m\}$ for which $cr(G_m) \neq oddcr(G_m)$, and even $\lim_{m\to\infty} \frac{oddcr(G_m)}{cr(G_m)} = \frac{\sqrt{3}}{2} \neq 1$. The conjectured equality is not just false, it's very very false.

4 Formula for $ioddcr(G)$: Proof sketch

While most crossing numbers are difficult to evaluate (indeed, no formula exists for calculating the crossing number, or even its asymptotics, for most graphs), $ioddcr(G)$ does indeed have a formula. This formula is difficult to evaluate, but the fact of its existence gives some hope for further discoveries.

The formula is a bit long, and the proof of it is very long. The original paper is 22 pages. Instead of giving a detailed proof, I will just outline it here.

First, many definitions. Let $\mathcal{B}$ be some set of partitions $\{\|_{xy}\colon xy \in E(G)\}$, where $\|_{xy}$ signifies a partition of $V(G) - \{x, y\}$. Write $u \|_{xy} z$ if u and z are in different sets of the partition $\|_{xy}$. Let $P_{\mathcal{B}}(xy, uz) = 1$ if $u \|_{xy} z$, and 0 otherwise. Fix some cyclic order $\mathcal{C} = v_1, v_2 \dots v_n$ of $V(G)$. Two edges xy and uz are said to be in acyclic order when the cyclic order $v_1 \dots v_n$ restricted to $\{x, y, u, z\}$ is x, u, y, z or x, z, y, u. Otherwise, they are in cyclic order. Let $O_{\mathcal{C}}(xy, uz) = 1$ if xy and uz are in cyclic order, and 0 otherwise. Now define

$$\begin{aligned} forced_{\mathcal{B},\mathcal{C}}(xy, uz) =& [1 - O_{\mathcal{C}}(xy, uz)][1 - P_{\mathcal{B}}(xy, uz)][1 - P_{\mathcal{B}}(uz, xy)] \\ &+ [1 - O_{\mathcal{C}}(xy, uz)]P_{\mathcal{B}}(xy, uz)P_{\mathcal{B}}(uz, xy) \\ &+ O_{\mathcal{C}}(xy, uz)[1 - P_{\mathcal{B}}(xy, uz)]P_{\mathcal{B}}(uz, xy) \\ &+ O_{\mathcal{C}}(xy, uz)P_{\mathcal{B}}(xy, uz)[1 - P_{\mathcal{B}}(uz, xy)] \end{aligned}$$

if $\{x, y\} \cap \{u, z\} = \emptyset$ and 0 otherwise. Then the formula for $ioddcr(G)$ is

$$ioddcr(G) = \min_{\mathcal{B}} \frac{1}{2} \sum_{xy \in E(G)} \sum_{uz \in E(G)} forced_{\mathcal{B},\mathcal{C}}(xy, uz)$$

This is proved by transforming a graph drawing into a set of closed curves on π^*, the one-point compactification of the plane π. Then a similar result is proved about crossings of these curves in π^*. Finally, the set of curves is transformed back to the graph drawing in the plane which minimizes $ioddcr(G)$.

Suppose we have a drawing of G in the plane which realizes $ioddcr(G)$. Transform it (without changing the crossings) so that the vertices are in cyclic order $v_1 \dots v_n$ around a circle $S \subset \pi$. Now extend a ray V_i out of and perpendicular to S from each v_i. Each edge e_{ij} between vertices v_i and v_j can now be identified with the closed curve ${e_{ij}}^* = V_i \cup e_{ij} \cup V_j \cup \infty$. Two such curves cross each other at ∞ if the vertices involved are in acyclic order. Otherwise, the curves are tangential at ∞. Also, this drawing defines an equivalence relation $\sim_e$ on the vertices for each edge e: $u \sim_e v$ iff a curve connecting u and v crosses e^* an even number of times. That is, $u \sim_e v$ iff they are both on the inside of e^* or both on the outside of e^*. This relation has only two equivalence classes, so it can be viewed as a bipartition of the vertices of G. Two functions O_S and P^* can be defined in this context analogously to $O_{\mathcal{C}}$ and $P_{\mathcal{B}}$. Furthermore, two independent edges xy and uz

cross an odd number of times in the original drawing of G when

$$\begin{aligned}CR(xy, uz) =&[1 - O_S(xy, uz)][1 - P^*(xy, uz)][1 - P^*(uz, xy)]\\ &+ [1 - O_S(xy, uz)]P^*(xy, uz)P^*(uz, xy)\\ &+ O_S(xy, uz)[1 - P^*(xy, uz)]P^*(uz, xy)\\ &+ O_S(xy, uz)P^*(xy, uz)[1 - P^*(uz, xy)]\end{aligned}$$

is 1, and an even number of times when $CR(xy, uz) = 0$. So we know that

$$ioddcr(G) \geq \min_{\mathcal{B}} \frac{1}{2} \sum_{xy \in E(G)} \sum_{uz \in E(G)} forced_{\mathcal{B},\mathcal{C}}(xy, uz)$$

since we've found a cyclic order and bipartition that achieves $ioddcr(G)$. On the other hand, given a minimizing bipartition and cyclic order, it's possible to construct a drawing achieving that independent crossing number: arrange the vertices in that cyclic order on a circle, and draw curves to represent edges such that they create the correct bipartition with their equivalence classes. So in fact

$$ioddcr(G) = \min_{\mathcal{B}} \frac{1}{2} \sum_{xy \in E(G)} \sum_{uz \in E(G)} forced_{\mathcal{B},\mathcal{C}}(xy, uz)$$

While this formula is still difficult to evaluate, and still involves minimizing over a large set, we no longer are minimizing over all possible graph drawings. Instead, we're dealing with a comparatively well-behaved set of familiar objects: permutations and partitions. Perhaps applying the wide range of knowledge we already have about these combinatorial objects will lead to advances in graph drawings.