18.319, Fall 2005. Problem Set 4

Problems from Matoušek's textbook on Discrete Geometry are marked with 🚲; hard (but feasible) problems are marked with $\star$.

1. (a) Prove that the maximum length of the Davenport-Schinzel sequence of order 2 over an alphabet of n letters is $\lambda_2(n) = 2n - 1$. 🚲
   (b) Show that for every n and s, $\lambda_s(n) \leq 1 + (s+1)\binom{n}{2}$. 🚲
   (c) Show that the lower envelope of n half-lines in the plane has $O(n)$ complexity. 🚲

2. Let $P_1, P_2, \ldots, P_m$ be convex polygons in the plane such that their vertex sets are disjoint (but the polygons are not necessarily disjoint). Assume that there are a total of n vertices and they are in general position. Show that the number of lines intersecting all polygons and tangent to exactly two of them is $O(\lambda_3(n))$. 🚲

3. Consider a cell C in an arrangement of n line segments in the plane. Let $|C|$ denote the complexity of the boundary of C.
   (a) Show that $|C| = O(\lambda_4(n))$. 🚲 (b) Show that $|C| = O(\lambda_3(n))$. $\star$ 🚲

4. You are given a function $\psi : \mathbb{N}^2 \to \mathbb{N}$ that satisfies $\psi(2,n) = 2n$ and the property that for every $p \in \mathbb{N}$, $1 \leq p \leq m$, there are $n_1, n_2 \in \mathbb{N}$ such that $n = n_1 + n_2$ and
$$\psi(m,n) \leq 4m + 4n_2 + \psi(p, n_2) + p \cdot \psi\left(\lceil m/p \rceil, \lfloor n_1/p \rfloor\right).$$
   (a) Prove that $\psi(2^j, n) \leq 4j \cdot 2^j + 6n$, for $j \geq 1$. (b) Show $\psi(2n, n) = O(n \log^* n)$. $\star$

5. Let $\mathrm{ex}(n, M)$ denote the maximum number of 1 entries in an $n \times n$ size 0-1 matrix that does not contain as submatrix any matrix of the family M. (a) (Füredi-Hajnal, 1992) $\mathrm{ex}(n, A) = \lambda_3(n) + O(n)$. $\star$ (b) (Füredi, Bienstock-Győri, 1990) $\mathrm{ex}(n, B) = \Theta(n \log n)$.
$$A = \begin{pmatrix} 1 & * & 1 & * \\ * & 1 & * & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} * & 1 & 1 \\ 1 & * & 1 \end{pmatrix}.$$

6. (a) Count the edges in the arrangement of n planes in general position in $\mathbb{R}^3$. 🚲
   (b) Express the number of k-dimensional faces in an arrangement of n hyperplanes in general position in $\mathbb{R}^d$ in terms of d, k, and n. 🚲
   (c) Prove that for every fixed $d \in \mathbb{N}$ the number of *unbounded* cells in an arrangement of n hyperplanes is $O(n^{d-1})$. 🚲

7. (Shrivastava) In the arrangement of n lines in the plane, consider the k-level and the $(2k)$-level for some $k \in \mathbb{N}$, $1 \leq k \leq n/2$. Show that there is a curve along the lines of the arrangement that separates these two levels and consists of $O(n/k)$ line segments.

8. (Sharir, 2001) Consider an arrangement of n lines in the plane. Let S be the largest subset of vertices such that none of the lines passes below more than k points of S.
   (a) Show that $|S| = O(n\sqrt{k})$. (b) Find an arrangement where $|S| = \Omega(n\sqrt{k})$.
   (c) For n points in general position in the plane, C is a set of circles such that each circle passes through three points and contains at most k points. Show that $|C| = O(nk^{2/3})$.