18.319, Fall 2005. Problem Set 2

Instructions: Solve your favourite problems from the list below. Open problems are marked with ✰; hard (but feasible) problems are marked with $\star$.

1. Let $a(P)$ denote the number of angles determined by ordered triples of a set P of non-collinear points in the plane. (We count angles $0^\circ \leq \angle(p_1, p_2, p_3) < 180^\circ$.) For $n \in \mathbb{N}$, $n \geq 3$, let $a(n) = \min_{|P|=n} a(P)$.
   (a) For every $n \in \mathbb{N}$, $n \geq 3$, find a set P_n of n points such that $a(P_n) = n - 2$.
   (b) Show that $a(n) = \Omega(n)$.
   (c) Prove or disprove that $a(n) = n - 2$. ✰
2. Prove the Sylvester-Gallai Theorem for a system of non-concurrent *pseudo-lines* in the plane: You are given a set of curves in the plane such that any two curves intersect in exactly one point, and no point is incident to all the curves. Show that there is a point incident to exactly two curves.
3. We are given n points and ℓ curves or surfaces in $\mathbb{R}^d$, $d \geq 2$. For any two real numbers $a, b > 1$, find two reals $e, f \in \mathbb{R}$ (in terms of a and b) such that

$$\#(k\text{−rich lines}) = O\left(\frac{n^a}{k^b}\right), \quad \forall k \in \mathbb{N} \quad \Leftrightarrow \quad \#(\text{incidences}) = O(n^e \ell^f).$$

4. (Elekes) Let X and Y be two sets of n real numbers ($X, Y \subset \mathbb{R}$, $|X| = |Y| = n$). Consider the Cartesian product $P = X \times Y = \{(x, y) \in \mathbb{R}^2 : x \in X, y \in Y\}$ in the plane. Show that the number of collinear triples of P is at most $O(n^4 \log n)$.
5. (Erdős) Consider n points in an integer lattice section $P = \{(a, b) \in \mathbb{N}^2 : 1 \leq a \leq \sqrt{n}, 1 \leq b \leq \sqrt{n}\}$ in the plane. Show that for every $\ell \in \mathbb{N}$, $\ell \geq \sqrt{n}$, there are ℓ lines in the plane such that the number of incidences with P is at least $\Omega(n^{2/3}\ell^{2/3} + n + \ell)$.
6. (Valtr, 2005) Let ∂B denote the boundary curve of a convex compact body B. Find a strictly convex compact body B in the plane with the following property: There are n translates of ∂B and n points in the plane such that the number of point-curve incidences is at least $\Omega(n^{4/3})$. $\star$
7. We are given n points and ℓ circles in the plane such that exactly x pairs of circles intersect. Show that the number of point-circle incidences is at most $O(n^{2/3}x^{1/3} + n + \ell)$.
8. Given a set S_n of n points in the plane, let $g(S_n)$ denote the number of *unit peremeter triangles* (that is, triangles where the sum of the three edge lengths is one).
   (a) Show that $g(S_n) = O(n^{7/3})$. (b) Show that $g(S_n) = O(n^{16/7})$. $\star$
9. (Hanani, 1934) If any two edges of a topological graph cross an even number of times, then the graph is planar. $\star$