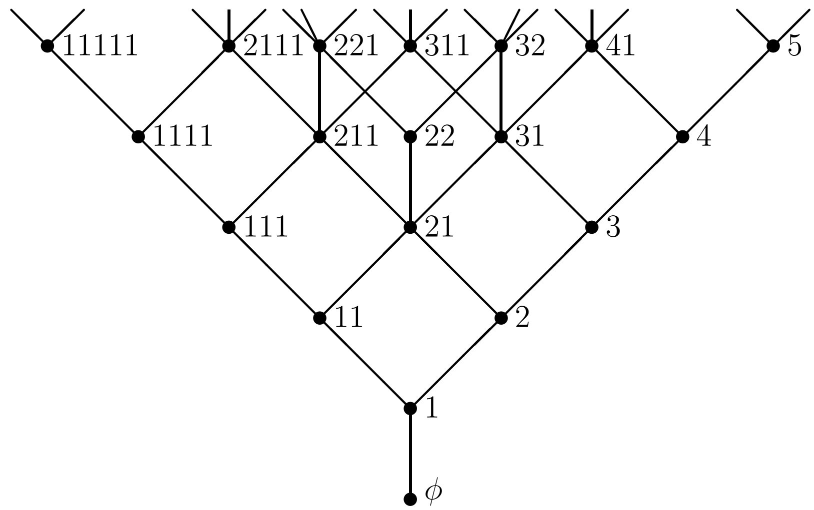
8 A glimpse of Young tableaux.

We defined in Section 6 Young's lattice Y, the poset of all partitions of all nonnegative integers, ordered by containment of their Young diagrams.



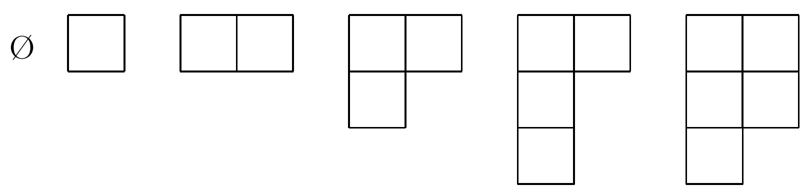
Young's lattice

Here we will be concerned with the counting of certain walks in the Hasse diagram (considered as a graph) of Y. Note that since Y is infinite, we cannot talk about its eigenvalues and eigenvectors. We need different techniques for counting walks. (It will be convenient to denote the length of a walk by n, rather than by ℓ as in previous sections.)

Note that Y is a graded poset (of infinite rank), with Y_i consisting of all partitions of i. In other words, we have $Y = Y_0 \cup Y_1 \cup \cdots$ (disjoint union), where every maximal chain intersects each level Y_i exactly once. We call Y_i the ith *level* of Y.

Since the Hasse diagram of Y is a simple graph (no loops or multiple edges), a walk of length n is specified by a sequence $\lambda^0, \lambda^1, \ldots, \lambda^n$ of vertices of Y. We will call a walk in the Hasse diagram of a poset a *Hasse walk*. Each λ^i is a partition of some integer, and we have either (a) $\lambda^i < \lambda^{i+1}$ and $|\lambda^i| = |\lambda^{i+1}| - 1$, or (b) $\lambda^i > \lambda^{i+1}$ and $|\lambda^i| = |\lambda^{i+1}| + 1$. A step of type (a) is denoted by U (for "up," since we move up in the Hasse diagram), while a step of type (b) is denoted by D (for "down"). If the walk W has steps of types $A_1, A_2, \ldots, A_n$, respectively, where each A_i is either U or D, then we say that W is of *type* $A_n A_{n-1} \cdots A_2 A_1$. Note that the type of a walk is written in the *opposite* order to that of the walk. This is because we will soon regard U and D as linear transformations, and we multiply linear transformations *right-to-left* (opposite to the usual left-to-right reading order). For instance (abbreviating a partition $(\lambda_1, \ldots, \lambda_m)$ as $\lambda_1 \cdots \lambda_m$), the walk $\emptyset, 1, 2, 1, 11, 111, 211, 221, 22, 21, 31, 41$ is of type $UUDDUUUUDUU = U^2D^2U^4DU^2$.

There is a nice combinatorial interpretation of walks of type U^n which begin at $\emptyset$. Such walks are of course just saturated chains $\emptyset = \lambda^0 < \lambda^1 < \cdots < \lambda^n$. In other words, they may be regarded as sequences of Young diagrams, beginning with the empty diagram and adding one new square at each step. An example of a walk of type U^5 is given by



We can specify this walk by taking the final diagram and inserting an i into square s if s was added at the ith step. Thus the above walk is encoded by the "tableau"

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

Such an object τ is called a *standard Young tableaux* (or SYT). It consists of the Young diagram D of some partition λ of an integer n, together with the numbers $1, 2, \ldots, n$ inserted into the squares of D, so that each number appears exactly once, and every row and column is *increasing*. We call λ the *shape* of the SYT τ, denoted $\lambda = \mathrm{sh}(\tau)$. For instance, there are five SYT of shape $(2,2,1)$, given by

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}\qquad\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}\qquad\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}\qquad\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}\qquad\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Let f^λ denote the number of SYT of shape λ, so for instance $f^{(2,2,1)} = 5$. The numbers f^λ have many interesting properties; for instance, there is a famous explicit formula for them known as the Frame-Robinson-Thrall hook formula. We will be concerned with their connection to counting walks in Young's lattice. If $w = A_n A_{n-1} \cdots A_1$ is some word in U and D and $\lambda \vdash n$, then let us write $\alpha(w, \lambda)$ for the number of Hasse walks in Y of type w which start at the empty partition $\emptyset$ and end at λ. For instance, $\alpha(UDUU, 11) = 2$, the corresponding walks being $\emptyset, 1, 2, 1, 11$ and $\emptyset, 1, 11, 1, 11$. Thus in particular $\alpha(U^n, \lambda) = f^\lambda$ [why?]. In a similar fashion, since the number of Hasse walks of type $D^n U^n$ which begin at $\emptyset$, go up to a partition $\lambda \vdash n$, and then back down to $\emptyset$ is given by $(f^\lambda)^2$, we have

$$\alpha(D^n U^n, \emptyset) = \sum_{\lambda \vdash n} (f^\lambda)^2. \tag{40}$$

Our object is to find an explicit formula for $\alpha(w, \lambda)$ of the form $f^\lambda c_w$, where c_w does not depend on λ. (It is by no means *a priori* obvious that such a formula should exist.) In particular, since $f^\emptyset = 1$, we will obtain by setting $\lambda = \emptyset$ a simple formula for the number of (closed) Hasse walks of type w from $\emptyset$ to $\emptyset$ (thus including a simple formula for (40)).

There is an easy condition for the existence of *any* Hasse walks of type w from $\emptyset$ to λ, given by the next lemma.

8.1 Lemma. *Suppose $w = D^{s_k} U^{r_k} \cdots D^{s_2} U^{r_2} D^{s_1} U^{r_1}$, where $r_i \geq 0$ and $s_i \geq 0$. Let $\lambda \vdash n$. Then there exists a Hasse walk of type w from $\emptyset$ to λ if and only if:*

$$\sum_{i=1}^{k} (r_i - s_i) = n$$

$$\sum_{i=1}^{j}(r_i - s_i) \geq 0 \text{ for } 1 \leq j \leq k.$$

Proof. Since each U moves up one level and each D moves down one level, we see that $\sum_{i=1}^{k}(r_i - s_i)$ is the level at which a walk of type w beginning at $\emptyset$ ends. Hence $\sum_{i=1}^{k}(r_i - s_i) = |\lambda| = n$.

After $\sum_{i=1}^{j}(r_i + s_i)$ steps we will be at level $\sum_{i=1}^{j}(r_i - s_i)$. Since the lowest level is level 0, we must have $\sum_{i=1}^{j}(r_i - s_i) \geq 0$ for $1 \leq j \leq k$.

The easy proof that the two conditions of the lemma are *sufficient* for the existence of a Hasse walk of type w from $\emptyset$ to λ is left to the reader. $\square$

If w is a word in U and D satisfying the conditions of Lemma 8.1, then we say that w is a *valid* λ*-word*. (Note that the condition of being a valid λ-word depends only on $|\lambda|$.)

The proof of our formula for $\alpha(w, \lambda)$ will be based on linear transformations analogous to those defined by (18) and (19). As in Section 4 let $\mathbb{R}Y_j$ be the real vector space with basis Y_j. Define two linear transformations $U_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i+1}$ and $D_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i-1}$ by

$$U_i(\lambda) = \sum_{\substack{\mu \vdash i+1 \\ \lambda < \mu}} \mu$$

$$D_i(\lambda) = \sum_{\substack{\nu \vdash i-1 \\ \nu < \lambda}} \nu,$$

for all $\lambda \vdash i$. For instance (using abbreviated notation for partitions)

$$U_{21}(54422211) = 64422211 + 55422211 + 54432211 + 54422221 + 544222111$$

$$D_{21}(54422211) = 44422211 + 54322211 + 54422111 + 5442221.$$

It is clear [why?] that if r is the number of *distinct* (i.e., unequal) parts of λ, then $U_i(\lambda)$ is a sum of $r + 1$ terms and $D_i(\lambda)$ is a sum of r terms. The next lemma is an analogue for Y of the corresponding result for B_n (Lemma 4.6).

8.2 Lemma. *For any $i \geq 0$ we have*

$$D_{i+1}U_i - U_{i-1}D_i = I_i, \tag{41}$$

the identity linear transformation on $\mathbb{R}Y_i$.

Proof. Apply the left-hand side of (41) to a partition λ of i, expand in terms of the basis Y_i, and consider the coefficient of a partition μ. If $\mu \neq \lambda$ and μ can be obtained from λ by adding one square s to (the Young diagram of) λ and then removing a (necessarily different) square t, then there is exactly one choice of s and t. Hence the coefficient of μ in $D_{i+1}U_i(\lambda)$ is equal to 1. But then there is exactly one way to remove a square from λ and then add a square to get μ, namely, remove t and add s. Hence the coefficient of μ in $U_{i-1}D_i(\lambda)$ is also 1, so the coefficient of μ when the left-hand side of (41) is applied to λ is 0.

If now $\mu \neq \lambda$ and we cannot obtain μ by adding a square and then deleting a square from λ (i.e., μ and λ differ in more than two rows), then clearly when we apply the left-hand side of (41) to λ, the coefficient of μ will be 0.

Finally consider the case $\lambda = \mu$. Let r be the number of distinct (unequal) parts of λ. Then the coefficient of λ in $D_{i+1}U_i(\lambda)$ is $r+1$, while the coefficient of λ in $U_{i-1}D_i(\lambda)$ is r, since there are $r+1$ ways to add a square to λ and then remove it, while there are r ways to remove a square and then add it back in. Hence when we apply the left-hand side of (41) to λ, the coefficient of λ is equal to 1.

Combining the conclusions of the three cases just considered shows that the left-hand side of (41) is just I_i, as was to be proved. $\Box$

We come to one of the main results of this section.

8.3 Theorem. *Let λ be a partition and $w = A_n A_{n-1} \cdots A_1$ a valid λ-word. Let $S_w = \{i : A_i = D\}$. For each $i \in S_w$, let a_i be the number of D's in w to the right of A_i, and let b_i be the number of U's in w to the right of A_i. Then*

$$\alpha(w, \lambda) = f^\lambda \prod_{i \in S_w} (b_i - a_i). \tag{42}$$

Before proving Theorem 8.3, let us give an example. Suppose $w = U^3D^2U^2DU^3 = UUUDDUUDUUU$ and $\lambda = (2,2,1)$. Then $S_w = \{4,7,8\}$ and $a_4 = 0$, $b_4 = 3$, $a_7 = 1$, $b_7 = 5$, $a_8 = 2$, $b_8 = 5$. We have also seen earlier that $f^{221} = 5$. Thus

$$\alpha(w,\lambda) = 5(3-0)(5-1)(5-2) = 180.$$

Proof of Theorem 8.3. Write $[\lambda]f$ for the coefficient of λ in $f \in \mathbb{R}Y_i$. We illustrate the proof for the special case $w = DU^\gamma DU^\beta DU^\alpha$, where $\alpha, \beta, \gamma \geq 0$, from which the general case will be clear. By the definition of w we have

$$\begin{aligned} \alpha(w,\lambda) &= [\lambda]w(\emptyset) \\ &= [\lambda]DU^\gamma DU^\beta DU^\alpha(\emptyset). \end{aligned}$$

We will use the identity (easily proved by induction on i)

$$DU^i = U^iD + iU^{i-1}. \tag{43}$$

Thus

$$\begin{aligned} w(\emptyset) &= DU^\gamma DU^\beta DU^\alpha(\emptyset) \\ &= DU^\gamma DU^\beta(U^\alpha D + \alpha U^{\alpha-1})(\emptyset) \\ &= \alpha DU^\gamma DU^{\alpha+\beta-1}(\emptyset), \end{aligned}$$

since $D(\emptyset) = 0$. Continuing, we obtain

$$\begin{aligned} w(\emptyset) &= \alpha DU^\gamma(U^{\alpha+\beta-1}D + (\alpha+\beta-1)U^{\alpha+\beta-2})(\emptyset) \\ &= \alpha(\alpha+\beta-1)DU^{\alpha+\beta+\gamma-2}(\emptyset) \\ &= \alpha(\alpha+\beta-1)(U^{\alpha+\beta+\gamma-2}D + (\alpha+\beta+\gamma-2)U^{\alpha+\beta+\gamma-3})(\emptyset) \\ &= \alpha(\alpha+\beta-1)(\alpha+\beta+\gamma-2)U^{\alpha+\beta+\gamma-3}(\emptyset). \end{aligned}$$

The coefficient of λ in $U^{\alpha+\beta+\gamma-3}(\emptyset)$ is f^λ, so we get

$$[\lambda]DU^\gamma DU^\beta DU^\alpha(\emptyset) = \alpha(\alpha+\beta-1)(\alpha+\beta+\gamma-2)f^\lambda,$$

which is equivalent to (42). $\square$

An interesting special case of the previous theorem allows us to evaluate equation (40).

8.4 Corollary. *We have*

$$\alpha(D^nU^n, \emptyset) = \sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

Proof. When $w = D^nU^n$ in Theorem 8.3 we have $S_w = \{n+1, n+2, \ldots, 2n\}$, $a_i = n - i$, and $b_i = n$, from which the proof is immediate. $\square$

NOTE (for those familiar with the representation theory of finite groups). It can be shown that the numbers f^λ, for $\lambda \vdash n$, are the degrees of the irreducible representations of the symmetric group $\mathcal{S}_n$. Given this, Corollary 8.4 is a special case of the result that the sum of the squares of the degrees of the irreducible representations of a finite group G is equal to the order $|G|$ of G. There are many other intimate connections between the representation theory of $\mathcal{S}_n$, on the one hand, and the combinatorics of Young's lattice and Young tableaux, on the other. There is also an elegant combinatorial proof of Corollary 8.4, known as the *Robinson-Schensted correspondence*, with many fascinating properties and with deep connections with representation theory.

We now consider a variation of Theorem 8.3 in which we are not concerned with the type w of a Hasse walk from $\emptyset$ to w, but only with the number of steps. For instance, there are three Hasse walks of length three from $\emptyset$ to the partition 1, given by $\emptyset, 1, \emptyset, 1$; $\emptyset, 1, 2, 1$; and $\emptyset, 1, 11, 1$. Let $\beta(\ell, \lambda)$ denote the number of Hasse walks of length ℓ from $\emptyset$ to λ. Note the two following easy facts:

(F1) $\beta(\ell, \lambda) = 0$ unless $\ell \equiv |\lambda| \pmod 2$.

(F2) $\beta(\ell, \lambda)$ is the coefficient of λ in the expansion of $(D+U)^\ell(\emptyset)$ as a linear combination of partitions.

Because of (F2) it is important to write $(D+U)^\ell$ as a linear combination of terms U^iD^j, just as in the proof of Theorem 8.3 we wrote a word w in U

and D in this form. Thus define integers $b_{ij}(\ell)$ by

$$(D+U)^\ell = \sum_{i,j} b_{ij}(\ell) U^i D^j. \tag{44}$$

Just as in the proof of Theorem 8.3, the numbers $b_{ij}(\ell)$ exist and are well-defined.

8.5 Lemma. *We have $b_{ij}(\ell) = 0$ if $\ell - i - j$ is odd. If $\ell - i - j = 2m$ then*

$$b_{ij}(\ell) = \frac{\ell!}{2^m\, i!\, j!\, m!}. \tag{45}$$

Proof. The assertion for $\ell - i - j$ odd is equivalent to (F1) above, so assume $\ell - i - j$ is even. The proof is by induction on ℓ. It's easy to check that (45) holds for $\ell = 1$. Now assume true for some fixed $\ell \geq 1$. Using (44) we obtain

$$\begin{aligned}
\sum_{i,j} b_{ij}(\ell+1) U^i D^j &= (D+U)^{\ell+1} \\
&= (D+U) \sum_{i,j} b_{ij}(\ell) U^i D^j \\
&= \sum_{i,j} b_{ij}(\ell)(DU^i D^j + U^{i+1} D^j).
\end{aligned}$$

In the proof of Theorem 8.3 we saw that $DU^i = U^i D + iU^{i-1}$ (see equation (43)). Hence we get

$$\sum_{i,j} b_{ij}(\ell+1) U^i D^j = \sum_{i,j} b_{ij}(\ell)(U^i D^{j+1} + iU^{i-1} D^j + U^{i+1} D^j). \tag{46}$$

As mentioned after (44), the expansion of $(D+U)^{\ell+1}$ in terms of $U^i D^j$ is unique. Hence equating coefficients of $U^i D^j$ on both sides of (46) yields the recurrence

$$b_{ij}(\ell+1) = b_{i,j-1}(\ell) + (i+1) b_{i+1,j}(\ell) + b_{i-1,j}(\ell). \tag{47}$$

It is a routine matter to check that the function $\ell!/2^m i! j! m!$ satisfies the same recurrence (47) as $b_{ij}(\ell)$, with the same intial condition $b_{00}(0) = 1$. From this the proof follows by induction. $\square$

From Lemma 8.5 it is easy to prove the following result.

8.6 Theorem. *Let $\ell \geq n$ and $\lambda \vdash n$, with $\ell - n$ even. Then*

$$\beta(\ell, \lambda) = \binom{\ell}{n}(1 \cdot 3 \cdot 5 \cdots (\ell - n - 1))f^{\lambda}.$$

Proof. Apply both sides of (44) to $\emptyset$. Since $U^i D^j(\emptyset) = 0$ unless $j = 0$, we get

$$\begin{aligned} (D+U)^{\ell}(\emptyset) &= \sum_i b_{i0}(\ell) U^i(\emptyset) \\ &= \sum_i b_{i0}(\ell) \sum_{\lambda \vdash i} f^{\lambda} \lambda. \end{aligned}$$

Since by Lemma 8.5 we have $b_{i0}(\ell) = \binom{\ell}{i}(1 \cdot 3 \cdot 5 \cdots (\ell - i - 1))$ when $\ell - i$ is even, the proof follows from (F2). □

NOTE. The proof of Theorem 8.6 only required knowing the value of $b_{i0}(\ell)$. However, in Lemma 8.5 we computed $b_{ij}(\ell)$ for all j. We could have carried out the proof so as only to compute $b_{i0}(\ell)$, but the general value of $b_{ij}(\ell)$ is so simple that we have included it too.

8.7 Corollary. *The total number of Hasse walks in Y of length $2m$ from $\emptyset$ to $\emptyset$ is given by*

$$\beta(2m, \emptyset) = 1 \cdot 3 \cdot 5 \cdots (2m - 1).$$

Proof. Simply substitute $\lambda = \emptyset$ (so $n = 0$) and $\ell = 2m$ in Theorem 8.6. □

The fact that we can count various kinds of Hasse walks in Y suggests that there may be some finite graphs related to Y whose eigenvalues we can also compute. This is indeed the case, and we will discuss the simplest case here. Let $Y_{j-1,j}$ denote the restriction of Young's lattice Y to ranks $j - 1$ and j. Identify $Y_{j-1,j}$ with its Hasse diagram, regarded as a (bipartite) graph. Let $p(i) = |Y_i|$, the number of partitions of i. (The function $p(i)$ has been extensively studied, beginning with Euler, though we will not discuss its fascinating properties here.)

8.8 Theorem. *The eigenvalues of $Y_{j-1,j}$ are given as follows: 0 is an eigenvalue of multiplicity $p(j) - p(j-1)$; and for $1 \le s \le j$, the numbers $\pm\sqrt{s}$ are eigenvalues of multiplicity $p(j-s) - p(j-s-1)$.*

Proof. Let $\boldsymbol{A}$ denote the adjacency matrix of $Y_{j-1,j}$. Since $\mathbb{R}Y_{j-1,j} = \mathbb{R}Y_{j-1} \oplus \mathbb{R}Y_j$ (vector space direct sum), any vector $v \in \mathbb{R}Y_{j-1,j}$ can be written uniquely as $v = v_{j-1} + v_j$, where $v_i \in \mathbb{R}Y_i$. The matrix $\boldsymbol{A}$ acts on the vector space $\mathbb{R}Y_{j-1,j}$ as follows [why?]:

$$\boldsymbol{A}(v) = D(v_j) + U(v_{j-1}). \tag{48}$$

Just as Theorem 4.7 followed from Lemma 4.6, we deduce from Lemma 8.2 that for any i we have that $U_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i+1}$ is one-to-one and $D_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i-1}$ is onto. It follows in particular that

$$\begin{aligned} \dim(\ker(D_i)) &= \dim \mathbb{R}Y_i - \dim \mathbb{R}Y_{i-1} \\ &= p(i) - p(i-1), \end{aligned}$$

where ker denotes kernel.

Case 1. Let $v \in \ker(D_j)$, so $v = v_j$. Then $\boldsymbol{A}v = Dv = 0$. Thus $\ker(D_j)$ is an eigenspace of $\boldsymbol{A}$ for the eigenvalue 0, so 0 is an eigenvalue of multiplicity at least $p(j) - p(j-1)$.

Case 2. Let $v \in \ker(D_s)$ for some $0 \le s \le j-1$. Let

$$v^* = \pm\sqrt{j-s}\,U^{j-1-s}(v) + U^{j-s}(v).$$

Note that $v^* \in \mathbb{R}Y_{j-1,j}$, with $v^*_{j-1} = \pm\sqrt{j-s}\,U^{j-1-s}(v)$ and $v^*_j = U^{j-s}(v)$. Using equation (43), we compute

$$\begin{aligned} \boldsymbol{A}(v^*) &= U(v^*_{j-1}) + D(v^*_j) \\ &= \pm\sqrt{j-s}\,U^{j-s}(v) + DU^{j-s}(v) \\ &= \pm\sqrt{j-s}\,U^{j-s}(v) + U^{j-s}D(v) + (j-s)U^{j-s-1}(v) \end{aligned}$$

$$
\begin{aligned}
&= \pm\sqrt{j-s}U^{j-s}(v) + (j-s)U^{j-s-1}(v) \\
&= \pm\left(\sqrt{j-s}\right)v^*. \qquad (49)
\end{aligned}
$$

It's easy to verify (using the fact that U is one-to-one) that if $v(1), \ldots, v(t)$ is a basis for $\ker(D_s)$, then $v(1)^*, \ldots, v(t)^*$ are linearly independent. Hence by (49) we have that $\pm\sqrt{j-s}$ is an eigenvalue of $\boldsymbol{A}$ of multiplicity at least $t = \dim(\ker(D_s)) = p(s) - p(s-1)$.

We have found a total of

$$
p(j) - p(j-1) + 2\sum_{s=0}^{j-1}(p(s) - p(s-1)) = p(j-1) + p(j)
$$

eigenvalues of $\boldsymbol{A}$. (The factor 2 above arises from the fact that both $+\sqrt{j-s}$ and $-\sqrt{j-s}$ are eigenvalues.) Since the graph $Y_{j-1,j}$ has $p(j-1) + p(j)$ vertices, we have found all its eigenvalues. □

An elegant combinatorial consequence of Theorem 8.8 is the following.

8.9 Corollary. *Fix $j \geq 1$. The number of ways to choose a partition λ of j, then delete a square from λ (keeping it a partition), then insert a square, then delete a square, etc., for a total of m insertions and m deletions, ending back at λ, is given by*

$$
\sum_{s=1}^{j}[p(j-s) - p(j-s-1)]s^m, \;\; m > 0. \qquad (50)
$$

Proof. Exactly half the closed walks in $Y_{j-1,j}$ of length $2m$ begin at an element of Y_j [why?]. Hence if $Y_{j-1,j}$ has eigenvalues $\theta_1, \ldots, \theta_r$, then by Corollary 1.3 the desired number of walks is given by $\frac{1}{2}(\theta_1^{2m} + \cdots + \theta_r^{2m})$. Using the values of $\theta_1, \ldots, \theta_r$ given by Theorem 8.8 yields (50). □

For instance, when $j = 7$, equation (50) becomes $4 + 2\cdot 2^m + 2\cdot 3^m + 4^m + 5^m + 7^m$. When $m = 1$ we get 30, the number of edges of the graph $Y_{6,7}$ [why?].