18.307: Integral Equations   Spring 2006

Handout ___ : Review of Fourier series and Fourier transforms

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Generalities:

(A) Fourier series (classical theory)

Recall that a "class" of functions that are periodic in $(-\infty,\infty)$ with period $b-a$ admit a series expansion in sines and cosines:

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot \cos\left(\frac{2n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{2n\pi x}{b-a}\right), \qquad a<x<b,$$

where

$$a_0 = \frac{1}{b-a}\int_a^b dx \cdot f(x), \qquad a_{n\neq 0} = \frac{2}{b-a}\int_a^b dx \cdot f(x)\cos\left(\frac{2n\pi x}{b-a}\right),$$

$$b_n = \frac{2}{b-a}\int_a^b dx \cdot f(x)\cdot \sin\left(\frac{2n\pi x}{b-a}\right). \qquad (b_0=0).$$

This expansion holds if $f(x)$ is square integrable in $[a,b]$:

$$\int_a^b dx \cdot |f(x)|^2 < \infty .$$

The convergence of the aforementioned series is then interpreted as "convergence in the mean," i.e.,

$$\lim_{N\to\infty} \int_a^b dx \cdot \left| f(x) - \sum_{n=0}^{N}\left[a_n \cdot \cos\left(\frac{2n\pi x}{b-a}\right) + b_n \cdot \sin\left(\frac{2n\pi x}{b-a}\right)\right]\right|^2 = 0.$$

This formula means that the RHS of the expansion for $f(x)$ may not converge to $f(x)$ pointwise, and hence is of different nature from, say, the Taylor expansion.

In particular, for the symmetric interval $[-L, L]$, where $a=-L$ and $b=L$, one gets

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + b_n \cdot \sin\left(\frac{n\pi x}{L}\right) \right],$$

where

$$a_0 = \frac{1}{2L} \int_a^b dx \cdot f(x), \qquad \left.\begin{matrix} a_{n\neq 0} \\ b_n \end{matrix}\right\} = \frac{1}{L} \int_a^b dx \cdot f(x) \cdot \left\{ \begin{matrix} \cos\left(\frac{n\pi x}{L}\right) \\ \sin\left(\frac{n\pi x}{L}\right) \end{matrix} \right. .$$

The last expression for $f(x)$ can be recast in an elegant form by replacing the sines and cosines by exponentials:

$$\cos\left(\frac{n\pi x}{L}\right) = \frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2}, \qquad \sin\left(\frac{n\pi x}{L}\right) = \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i}.$$

Accordingly,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2}\right) \cdot e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2}\right) \cdot e^{-i\frac{n\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L},$$

by changing the summation variable from n to $-n$ in the second sum. In the above,

$$c_n = \begin{cases} a_0, & n = 0 \\ \dfrac{a_{|n|} \mp i\, b_{|n|}}{2}, & n \gtrless 0. \end{cases}$$

$$\Rightarrow \quad c_n = \frac{1}{2L} \int_{-L}^{L} dx \cdot f(x)\, e^{-in\pi x/L}, \qquad \text{all } n.$$

(B) Fourier transform

We start with the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L} \quad , \qquad x \text{ in } (-L, L),$$

and we eventually allow $L \to \infty$ in order to find a representation for $f(x)$ in $(-\infty, \infty)$. For this purpose, consider the quantity

$$\omega_n = \frac{n\pi}{L} \qquad \Rightarrow \Delta\omega_n \equiv \omega_{n+1} - \omega_n = \frac{\pi}{L} \to 0 \quad \text{as} \quad L \to +\infty.$$

Hence, in the limit $L \to \infty$ ω_n can be treated as a continuous variable.
The coefficients c_n are

$$c_n = \frac{1}{2L} \int_{-L}^{L} dx \; f(x) \; e^{-in\pi x/L}$$

$$\rightarrow \quad (c_n \cdot 2L) = \int_{-L}^{L} dx \; f(x) \; e^{-i\omega_n x}$$

If we <u>assume</u> that the integral in RHS <u>converges</u> as $L \to \infty$, then (by treating $\omega_n \equiv \omega$ as a <u>continuous</u> variable) we get

$$\lim_{L\to\infty} (c_n \cdot 2L) = \int_{-\infty}^{\infty} dx \; f(x) \, e^{-i\omega x} \equiv \tilde{f}(\omega) \; : \; \text{Fourier transform (FT) of } f(x) \text{ in } (-\infty, \infty)$$

The function $f(x)$ is recovered as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L} = \sum_{n=-\infty}^{\infty} (c_n \cdot 2L) \frac{1}{2L} e^{in\pi x/L} = \sum_{n=-\infty}^{\infty} \frac{\tilde{f}(\omega_n)}{2L} e^{i\omega_n x}$$

$$= \sum_{n=-\infty}^{\infty} \tilde{f}(\omega_n) \cdot e^{i\omega_n x} \cdot \left(\frac{\Delta\omega_n}{2\pi}\right) \xrightarrow[L\to\infty]{} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \tilde{f}(\omega) \, e^{i\omega x} \; : \; \text{"inverse F.T."}$$

Particulars of the Fourier transform :

In most part of this course we define the F.T. of a function $f(x)$, where $-\infty < x < \infty$, as

$$\tilde{f}(\xi) = \int_{-\infty}^{\infty} dx \ f(x) \ e^{-i\xi x} . \qquad (1)$$

(I will use mostly ξ or k instead of ω as the Fourier variable).
There are two distinct theories of Fourier transforms: (Now you must forget how we derived the formulas for the FT on p. 3 since the discussion there was heuristic.)

(I) Fourier transform of square integrable functions.

It is assumed that
$$\int_{-\infty}^{\infty} dx \ |f(x)|^2 < \infty . \qquad (2)$$

The inverse F.T for (1) is given by

$$f(x) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \ e^{i\xi x} \ \tilde{f}(\xi) . \qquad (3)$$
[Note: Don't forget the $(2\pi)^{-1}$.]

Note that in this case $\tilde{f}(\xi)$ is defined for real ξ. Accordingly, the path of integration in (3) coincides with the entire real axis, i.e., all ξ's – over which we integrate – are real.
Equation (1) is meaningful in the sense of "convergence in the mean," i.e., (1) means that there exists $\tilde{f}(\xi)$ for all real ξ such that

$$\lim_{R\to+\infty} \int_{-\infty}^{\infty} d\xi \ \left| \tilde{f}(\xi) - \int_{-R}^{R} dx \ f(x) \ e^{-i\xi x} \right|^2 = 0 . \qquad (4)$$

Symbolically, one writes

$$\tilde{f}(\jmath) = \underset{R\to+\infty}{\text{l.i.m.}} \int_{-R}^{R} dx \;\; f(x) \;\; e^{-i\jmath x}. \tag{5}$$

Similarly, Eq. (3) means that, given $\tilde{f}(\jmath)$, there exists an $f(x)$ such that

$$\lim_{R\to+\infty} \int_{-\infty}^{+\infty} dx \cdot \left| f(x) - \int_{-R}^{R} \frac{d\jmath}{2\pi} \;\; e^{i\jmath x} \;\; \tilde{f}(\jmath) \right|^2 = 0. \tag{6}$$

It can then be proved that

$$\int_{-\infty}^{\infty} d\jmath \;\; |\tilde{f}(\jmath)|^2 = 2\pi \int_{-\infty}^{+\infty} dx \;\; |f(x)|^2, \tag{7}$$

which is Parseval's identity for square integrable functions.
We see that the pair $(f(x), \tilde{f}(\jmath))$ defined this way consists of two functions with very similar properties. This situation may change drastically if condition (2) is relaxed.

Ⅱ Fourier transforms of integrable functions

The condition on $f(x)$ reads as

$$\int_{-\infty}^{+\infty} dx \cdot |f(x)| < \infty. \tag{8}$$

Then $\tilde{f}(\jmath)$ is still defined for <u>real</u> $\jmath$. Indeed, from (1) one gets

$$|\tilde{f}(\jmath{:}\text{real})| = \left| \int_{-\infty}^{+\infty} dx \;\; f(x)\, e^{-i\jmath x} \right| \le \int_{-\infty}^{+\infty} dx \;\; |f(x) \cdot e^{-i\jmath x}| = \int_{-\infty}^{+\infty} dx \cdot |f(x)| < \infty. \tag{9}$$

One can further show that the function

$$\tilde{f}_{+}(\jmath) = \int_{-\infty}^{0} dx \;\; e^{-i\jmath x} \;\; f(x) \tag{10}$$

is analytic in the <u>upper</u> half of the ζ plane, while

$$\tilde{f}_+(\zeta) \to 0 \quad \text{as} \quad |\zeta| \to \infty \quad \text{with} \quad \text{Im}\,\zeta > 0. \tag{11}$$

Similarly, the function

$$\tilde{f}_-(\zeta) = \int_0^{\infty} dx \quad e^{-i\zeta x} \quad f(x) \tag{12}$$

is analytic in the <u>lower</u> half plane and

$$\tilde{f}_-(\zeta) \to 0 \quad \text{as} \quad |\zeta| \to \infty \quad \text{with } \text{Im}\,\zeta < 0.$$

Clearly,

$$\tilde{f}(\zeta) = \tilde{f}_+(\zeta) + \tilde{f}_-(\zeta) \quad , \quad \zeta : \text{real}. \tag{13}$$

It can be shown that

$$\tilde{f}(\zeta) \to 0 \quad \text{as} \quad \zeta \to \pm\infty \quad (\zeta : \text{real}), \tag{14}$$

a property <u>in common</u> with FT's of square integrable functions.

<u>Example</u> Solve the boundary-value problem $\begin{cases} u''(x) - \lambda^2 u(x) = f(x) \quad (\lambda > 0) \\ u(x \to \pm\infty) = 0 \;, \; -\infty < x < \infty \\ (\text{"sufficiently fast"}) \end{cases}$

by use of the Fourier transform.

<u>Solution</u>: The FT of $u(x)$ is (I use $k \equiv \zeta$)

$$\tilde{u}(k) = \int_{-\infty}^{\infty} dx \quad e^{-ikx} \quad u(x) \tag{15}$$

By taking the FT of both sides of the given ODE, we obtain

$$\underbrace{\int_{-\infty}^{\infty} dx\; e^{-ikx}\, u''(x)}_{\text{(integration by parts)}} - \lambda^2 \underbrace{\int_{-\infty}^{\infty} dx\; e^{-ikx}\, u(x)}_{\tilde{u}(k)} = \underbrace{\int_{-\infty}^{\infty} dx\; e^{-ikx}\, f(x)}_{\tilde{f}(k):\ \text{FT of } f(x)\ \text{(given)}}$$

$$\left\{ \begin{array}{l} e^{-ikx}\, u'(x)\Big|_{x=-\infty}^{\infty} + ik\, e^{-ikx} u(x)\Big|_{x=-\infty}^{\infty} \\ - k^2 \displaystyle\int_{-\infty}^{\infty} dx\; e^{-ikx}\, u(x) \end{array} \right.$$

By assuming that $u(x \to \pm\infty) = 0$ <u>and</u> $u'(x \to \pm\infty) = 0$ the last equations give an algebraic equation for $\tilde{u}(k)$:

$$-(k^2 + \lambda^2)\, \tilde{u}(k) = \tilde{f}(k) \quad \Rightarrow \quad \tilde{u}(k) = -\frac{\tilde{f}(k)}{k^2 + \lambda^2} \tag{16}$$

Hence,

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\; e^{ikx}\, \tilde{u}(k) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} dk\; e^{ikx}\, \frac{\tilde{f}(k)}{k^2 + \lambda^2}$$

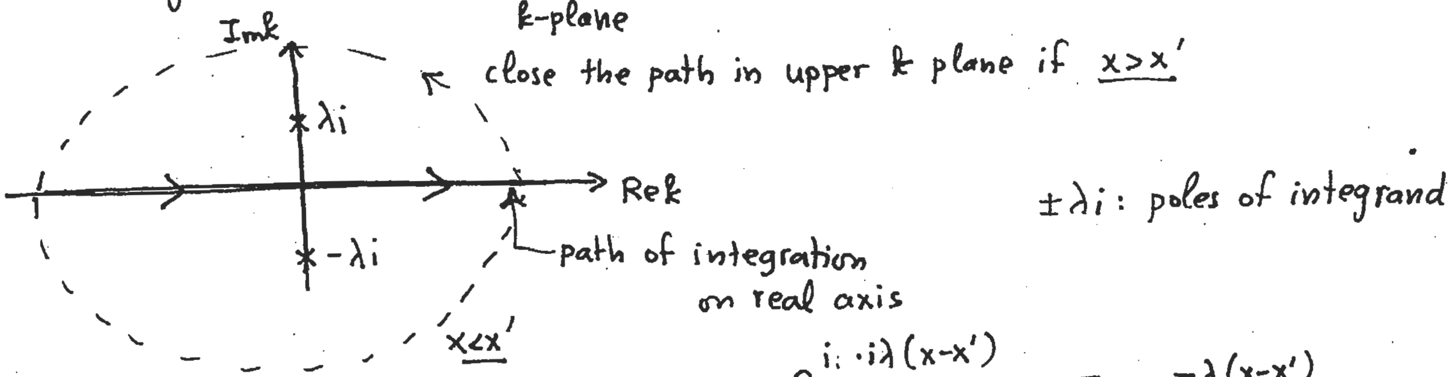
We may further simplify this formula using

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx'\; e^{-ikx'}\, f(x').$$

Then,

$$u(x) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} dx'\, f(x') \int_{-\infty}^{\infty} dk\; \frac{e^{ik(x-x')}}{k^2 + \lambda^2} \tag{17}$$

The integral in k can be evaluated by various means (for example, by contour integration, as described below).



$$\int_{-\infty}^{\infty} dk \; \frac{e^{ik(x-x')}}{k^2+\lambda^2} = \begin{cases} \underline{x>x'}: \; 2\pi i \cdot \dfrac{e^{i\cdot i\lambda(x-x')}}{2i\lambda} = \dfrac{\pi}{\lambda} \; e^{-\lambda(x-x')} \\[2ex] \underline{x<x'}: \; -2\pi i \; \dfrac{e^{i(-i\lambda)\cdot(x-x')}}{-2i\lambda} = \dfrac{\pi}{\lambda} \; e^{\lambda(x-x')} \end{cases} \qquad (18)$$

via applying the Cauchy integral formula.

Finally, from (17)

$$u(x) = -\frac{1}{2\lambda} \int_{-\infty}^{\infty} dx' \; f(x') \; e^{-\lambda\cdot|x-x'|} \, . \qquad (19)$$

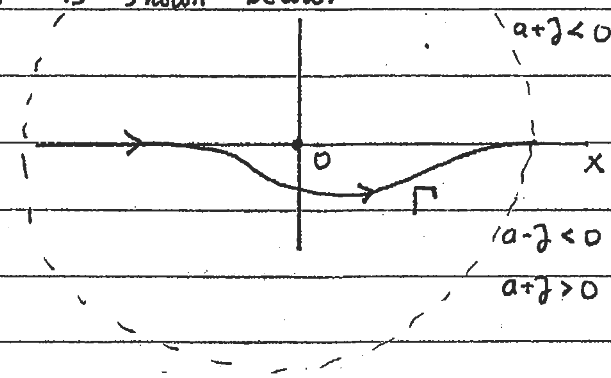
Examples of evaluating Fourier Transforms of functions via contour integration:

Example 1 $\qquad f(x) = \dfrac{\sin(ax)}{x}, \quad a>0, \quad -\infty < x < +\infty.$ (15)

The FT of this function is

$$\tilde{f}(\lambda) = \int_{-\infty}^{+\infty} dx \, \frac{\sin(ax)}{x} e^{-i\lambda x} = \frac{1}{2i} \int_{-\infty}^{+\infty} dx \, \frac{e^{i(a-\lambda)x} - e^{-i(a+\lambda)x}}{x}$$

$$= \frac{1}{2i} \int_{\Gamma} d\cancel{z} \, \frac{e^{i(a-\lambda)\cancel{z}} - e^{-i(a+\lambda)\cancel{z}}}{\cancel{z}}$$ where Γ is shown below.



Γ is introduced in order to avoid the singularity (pole) at $x=0$ by breaking the integral into two parts.

$$\tilde{f}(\mathfrak{z}) = \frac{1}{2i}\int_\Gamma dx\ \frac{e^{i(a-\mathfrak{z})x}}{x} - \frac{1}{2i}\int_\Gamma dx\ \frac{e^{-i(a+\mathfrak{z})x}}{x} = \begin{cases} \pi, & |\mathfrak{z}|<a \\ 0, & |\mathfrak{z}|>a. \end{cases} \quad (16)$$

where each integral is evaluated by contour integration. Note that

$\tilde{f}(\mathfrak{z}=\pm a) = \frac{\pi}{2}$, which is equal to $\frac{1}{2}\left[\tilde{f}(\mathfrak{z}=\pm a^+) + \tilde{f}(\mathfrak{z}=\pm a^-)\right]$.

Example 2 $\quad f(x) = \dfrac{\sin(ax)}{x(x^2+b^2)}, \quad a,b>0, \quad -\infty<x<+\infty. \quad (17)$

[Note that $f(x) = O\left(\frac{1}{x^3}\right)$ as $x\to\pm\infty$, whereas in Ex. 2 $f(x) = O\left(\frac{1}{x}\right)$ as $x\to+\infty$.]

$$\tilde{f}(\mathfrak{z}) = \frac{1}{2i}\int_\Gamma dz\ \frac{e^{i(a-\mathfrak{z})x}}{z(z^2+b^2)} - \frac{1}{2i}\int_\Gamma dz\ \frac{e^{-i(a+\mathfrak{z})x}}{z(z^2+b^2)}. \quad (18)$$

$$= \begin{cases} \frac{\pi}{b^2}\sinh(ab)\, e^{b\mathfrak{z}}, & \mathfrak{z}<-a, \\ \frac{\pi}{b^2}\left[1-e^{-ab}\cosh(b\mathfrak{z})\right], & |\mathfrak{z}|<a, \\ \frac{\pi}{b^2}\sinh(ab)\, e^{-b\mathfrak{z}}, & \mathfrak{z}>a. \end{cases} \quad (19)$$

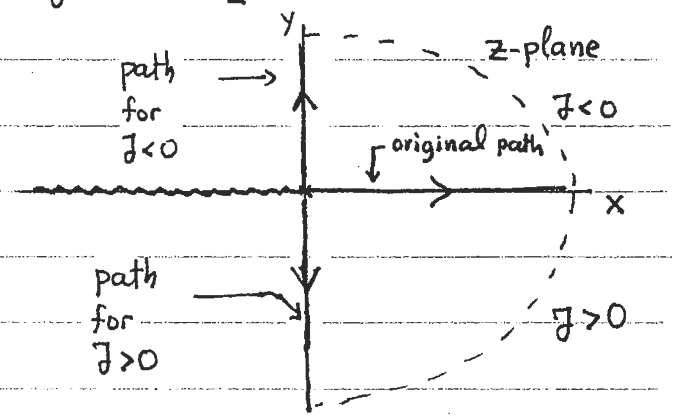
Notably, $\tilde{f}(\mathfrak{z})$ is continuous, and $\tilde{f}'(\mathfrak{z})$ is continuous ($\mathfrak{z}$: real).
However, $\tilde{f}''(\mathfrak{z})$ has step discontinuities at $\mathfrak{z}=\pm a$.
We see that the "rate" by which $f(x)\to 0$ as $|x|\to+\infty$ affects the degree of smoothness of $\tilde{f}(\mathfrak{z})$. For square integrable functions,

$f(x) = O\left(\frac{1}{x}\right)$ as $|x|\to\infty$ usually means $\tilde{f}(\mathfrak{z})$: step-discontinuous ($\mathfrak{z}$: real)
$f(x) = O\left(\frac{1}{x^2}\right)$ " " " $\tilde{f}(\mathfrak{z})$: continuous, $\tilde{f}'(\mathfrak{z})$: step-disc.
$f(x) = O\left(\frac{1}{x^3}\right)$ " " " $\tilde{f}(\mathfrak{z}), \tilde{f}'(\mathfrak{z})$: " , $\tilde{f}''(\mathfrak{z})$: " etc.

Example 3 $\qquad f(x) = \begin{cases} 0, & x<0 \\ x^{-1/2}, & x>0 \end{cases}$. $\qquad$ (20)

With the understanding that $f(x)>0$ for $x>0$, we may take a branch cut that lies in the negative real axis and define the first (corresponding) Riemann sheet as the one in which

$$-\pi < \operatorname{Arg} z \le \pi\ ,\quad \text{i.e.},\quad -\frac{\pi}{2} < \operatorname{Arg} f(z) \le \frac{\pi}{2}\,. \qquad (21)$$



For $\zeta>0$, one has to close the path (originally in $\{x>0\}$) in the lower half-plane.

For $\zeta<0$, one has to close the path by a large quarter of a circle in the upper half-plane.

$\zeta<0$:

$$\tilde{f}(\zeta) = \int_0^{+\infty} dx\ \frac{1}{\sqrt{x}}\ e^{-i\zeta x} = \int_0^{+i\infty} dz\ \frac{1}{\sqrt{z}}\ e^{-i\zeta z}$$

$$\underset{(y>0)}{\overset{z=iy}{=}} \int_0^{+\infty} d(iy)\ \frac{1}{[e^{i\pi/2}y]^{1/2}}\ e^{y\zeta} = e^{i\pi/4}\int_0^{+\infty} \frac{dy}{\sqrt{y}}\ e^{-y(-\zeta)}$$

$$= e^{i\pi/4}\int_0^{+\infty} \frac{dy}{\sqrt{y}}\ e^{-y|\zeta|} = \sqrt{\frac{\pi}{2|\zeta|}}\ (1+i). \qquad (21a)$$

$\zeta>0$

$$\tilde{f}(\zeta) = \int_0^{+\infty} dx\ \frac{1}{\sqrt{x}}\ e^{-i\zeta x} = \int_0^{-i\infty} dz\ \frac{1}{\sqrt{z}}\ e^{-i\zeta z}$$

$$\underset{y>0}{\overset{z=-iy}{=}} \int_0^{+\infty} d(-iy)\ \frac{1}{[e^{-i\pi/2}y]^{1/2}}\ e^{-\zeta y} = e^{-i\pi/4}\int_0^{+\infty} \frac{dy}{\sqrt{y}}\ e^{-\zeta y} = \sqrt{\frac{\pi}{2\zeta}}\ (1-i) \qquad (21b)$$

It follows that

$$\tilde{f}(\jmath) = \sqrt{\frac{\pi}{2\cdot|\jmath|}}\left(1 - i\,\frac{\jmath}{|\jmath|}\right), \qquad \jmath : \text{real}, \quad \jmath \neq 0. \tag{21}$$

Note that $\tilde{f}(\jmath)$ is <u>not</u> defined for $\jmath = 0$. This is not surprising, since $f(x)$ is neither square integrable (at $x=0$) nor integrable (as $x \to +\infty$). Inversion of this $\tilde{f}(\jmath)$ gives ($\tilde{f}(\jmath)$ is integrable at $\jmath = 0$ although it has a singularity there)

$$f(x) = (1-i)\int_0^{+\infty}\frac{d\jmath}{2\pi}\, e^{i\jmath x}\sqrt{\frac{\pi}{2\jmath}} + (1+i)\int_{-\infty}^{0}\frac{d\jmath}{2\pi}\, e^{i\jmath x}\sqrt{\frac{\pi}{-2\jmath}}$$

$$= \frac{1}{\pi}\sqrt{\frac{\pi}{2}}\, \mathrm{Re}\left[(1+i)\int_0^{+\infty} d\jmath\, \frac{e^{-i\jmath x}}{\sqrt{\jmath}}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\cdot \mathrm{Re}\left[(1+i)\cdot\begin{cases}(1-i)\sqrt{\frac{\pi}{2x}}, & x>0\\ (1+i)\sqrt{\frac{\pi}{2x}}, & x<0\end{cases}\right] = \begin{cases}\frac{1}{\sqrt{x}}, & x>0\\ 0, & x<0\end{cases}, \tag{22}$$

by using the result of the original integration that led to (21).

Let us try to interpret (21). Can this formula be thought of as stemming from an analytic function? It does not seem to imply so. To make this formula more transparent, consider

$$f_\alpha(x) = e^{-\alpha x}\cdot f(x) = \begin{cases}0, & x<0\\ e^{-\alpha x}\frac{1}{\sqrt{x}}, & x>0\end{cases}, \qquad \alpha > 0. \tag{23}$$

This $f_\alpha(x)$ is now integrable in $(-\infty, +\infty)$.

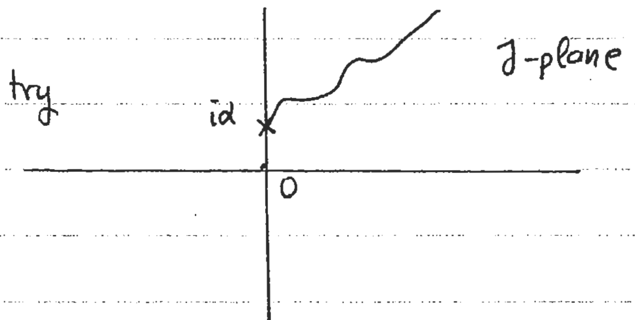
$$\tilde{f}_\alpha(\jmath) = \int_0^{+\infty} dx\, \frac{e^{-\alpha x}}{\sqrt{x}}\, e^{-i\jmath x} = \int_0^{+\infty} dx\, \frac{e^{-(\alpha+i\jmath)x}}{\sqrt{x}} = \frac{1}{\sqrt{\alpha+i\jmath}}\int_0^{\infty} dt\, \frac{e^{-t}}{\sqrt{t}}$$

$$\rightarrow \qquad \tilde{f}_\alpha(\zeta) = \sqrt{\frac{\pi}{\alpha + i\zeta}}\,, \qquad -\infty < \zeta < +\infty. \tag{24}$$

This $\tilde{f}_\alpha(\zeta)$ is defined for <u>all</u> real ζ. Furthermore, it is analytic in the <u>lower</u> half of the ζ-plane. This $\tilde{f}_\alpha(\zeta)$ is written as

$$\tilde{f}_\alpha(\zeta) = e^{-i\pi/4} \cdot \sqrt{\frac{\pi}{\zeta - i\alpha}}\,. \tag{25}$$

Because $\tilde{f}_\alpha(\zeta)$ is multi-valued, one needs to define a branch cut.



Since $f(x) = \lim_{\alpha \to 0^+} f_\alpha(x)$, it is reasonable to try to take the limit $\lim_{\alpha \to 0^+} \tilde{f}_\alpha(\zeta)$:

$$\lim_{\alpha \to 0^+} \tilde{f}_\alpha(\zeta) = \lim_{\alpha \to 0^+} \left[e^{-i\pi/4} \sqrt{\frac{\pi}{\zeta - i\alpha}} \right]. \tag{26}$$

For <u>$\zeta > 0$</u>, this limit gives $\lim_{\alpha \to 0^+} \tilde{f}_\alpha(\zeta) = e^{-i\pi/4} \sqrt{\frac{\pi}{\zeta}} = (1-i)\sqrt{\frac{\pi}{2\zeta}}$, (26b)

in agreement with (21b).

For <u>$\zeta < 0$</u>, we take $\zeta = |\zeta|\, e^{-i\pi}$ and the limit (26) gives

$$\lim_{\alpha \to 0^+} \tilde{f}_\alpha(\zeta) = e^{-i\pi/4} \sqrt{\frac{\pi}{e^{-i\pi}|\zeta|}} = e^{i\pi/4} \sqrt{\frac{\pi}{|\zeta|}} = (1+i)\sqrt{\frac{\pi}{2 \cdot |\zeta|}}\,, \tag{26a}$$

in agreement with (21a).

<u>Conclusion</u>: Eq. (21) can be viewed as the limit of (24) as $\alpha \to 0^+$.

In this sense, it defines the limit of a function that is analytic in the lower ζ-plane.

<u>Example 4</u>: $\quad f(x) = \begin{cases} x^m; & x>0 \\ 0, & x<0 \end{cases}, \quad m=0,1,2,\ldots \qquad (27)$

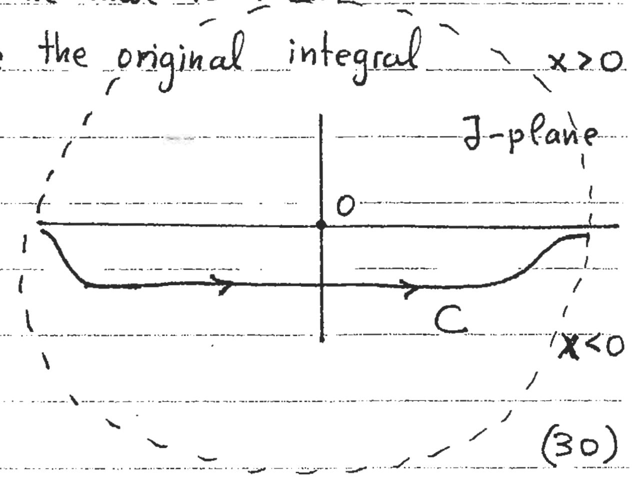
This $f(x)$ is neither integrable nor square integrable in $(-\infty,+\infty)$. In fact, its FT is <u>not</u> defined for any real ζ. By taking $\zeta = \xi - i\eta$ (ξ,η: real, $\eta>0$) consider the FT for ζ: complex, $\operatorname{Im}\zeta<0$:

$$\tilde{f}(\zeta) = \int_{-\infty}^{+\infty} dx \; e^{-i\zeta x} f(x) = \int_0^{+\infty} dx \; e^{-i\zeta x} x^m = \int_0^{+\infty} dx \; e^{-i\xi x} e^{-\eta x} x^m \neq \infty \quad \text{for } \eta>0. \quad (28)$$

$$\tilde{f}(\zeta) = i^m \frac{d^m}{d\zeta^m} \int_0^{+\infty} dx \; e^{-i\zeta x}, \qquad \operatorname{Im}\zeta<0,$$

$$= i^m \frac{d^m}{d\zeta^m} \frac{1}{i\zeta} = \frac{i^m}{i} \frac{(-1)^m m!}{\zeta^{m+1}} = \frac{1}{i^{m+1}} \frac{m!}{\zeta^{m+1}}. \qquad (29)$$

Clearly, $\tilde{f}(\zeta)$ is analytic in the lower ζ-plane, with an $m+1$-order pole at $\zeta=0$. $\tilde{f}(\zeta)$ is also analytic in the upper ζ-plane. Without any additional condition, it is ambiguous where we should take the inversion path in order to recover $f(x)$. The original condition <u>$\operatorname{Im}\zeta<0$</u> serves presisely this purpose. Hence, we have to invert $\tilde{f}(\zeta)$ in that part of the ζ-plane where the original integral $\tilde{f}(\zeta)$ made sense: $\operatorname{Im}\zeta<0$.



$$f(x) = \int_{C_1} \frac{d\zeta}{2\pi} e^{i\zeta x} \tilde{f}(\zeta)$$

$$= \int_{C_1} \frac{d\zeta}{2\pi} e^{i\zeta x} \frac{m!}{i^{m+1}} \frac{1}{\zeta^{m+1}}. \qquad (30)$$

Evidently,

$$f(x) = \begin{cases} 0, & x<0 \quad \text{(by closing the path in the lower } J\text{-plane)} \\ 2\pi i \dfrac{1}{2\pi} \operatorname*{Res}_{J=0} \left[\dfrac{m!}{i^{m+1}} \dfrac{e^{iJx}}{J^{m+1}} \right], & x>0, \end{cases} \qquad (31)$$

where

$$\operatorname*{Res}_{J=0} \left[\frac{m!}{i^{m+1}} \frac{e^{iJx}}{J^{m+1}} \right] = \frac{m!}{i^{m+1}} \operatorname*{Res}_{J=0} \frac{\sum_{n=0}^{\infty} \frac{(iJx)^n}{n!}}{J^{m+1}} = \frac{m!}{i^{m+1}} \operatorname*{Res}_{J=0} \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \frac{1}{J^{m-n+1}}$$

$$= \frac{m!}{i^{m+1}} \operatorname*{Res}_{J=0} \left[\frac{(ix)^m}{m!} \frac{1}{J} \right] = \frac{x^m}{i}. \qquad (32)$$

Hence,

$$f(x) = \begin{cases} 0, & x<0 \\ x^m, & x>0. \end{cases} \qquad (27)$$

An alternative way to think about the inversion path is to consider the function

$$\tilde{g}(\xi) = \tilde{f}(\xi - i\eta), \qquad \eta>0, \quad \eta: \underline{\text{fixed}}, \qquad (33)$$

which is wiewed as a function of the <u>real</u> variable ξ <u>only</u>. The inverse FT of this $\tilde{g}(\xi)$ is

$$g(x) = f(x)\, e^{-\eta x} = \begin{cases} 0, & x<0 \\ x^m e^{-\eta x}, & x>0 \end{cases}. \qquad (34)$$

Clearly, $g(x) \to 0$ as $|x| \to +\infty$ and $g(x)$ is square integrable, and integrable, in $(-\infty, +\infty)$. Then,

$$\tilde{g}(\xi) = \int_{-\infty}^{+\infty} dx\, g(x)\, e^{-i\overset{\text{real}}{\xi} x} = \int_{-\infty}^{+\infty} dx\, [f(x)\, e^{-\eta x}]\, e^{-i\xi x} = \int_{-\infty}^{+\infty} dx\, f(x)\, e^{-i(\xi - i\eta)x}$$

$$= \int_{-\infty}^{+\infty} dx\, f(x)\, e^{-iJx}, \qquad J = \xi - i\eta, \qquad \eta>0. \qquad (35)$$

The inversion formula then gives

$$g(x)=f(x)\,e^{-\eta x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \;\; \tilde{g}(\xi)\; e^{i\xi x}$$

$$\Rightarrow \quad f(x) = \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi} \; \underbrace{\tilde{g}(\xi)}_{\tilde{f}(\lambda)} \; e^{i\overset{\lambda=\xi-i\eta}{\lambda} x} = \int_{-\infty-i\eta}^{+\infty-i\eta} \frac{d(\xi-i\eta)}{2\pi}\; \tilde{f}(\xi-i\eta)\, e^{i\lambda x} = \int_{-\infty-i\eta}^{+\infty-i\eta} \frac{d\lambda}{2\pi}\; \tilde{f}(\lambda)\, e^{i\lambda x}. \quad (36)$$

Similarly, inversion paths that are used to calculate functions $f(x)$ with the property $f(x>0)=0$ lie in the <u>upper</u> λ-plane because the respective FT's, $\tilde{f}(\lambda)$, are analytic in the upper λ-plane.

More generally, let $f(x)$ be any (admissible) function defined for $-\infty < x < +\infty$ and take $f(x)=f_1(x)+f_2(x)$, where

$$f_1(x)=0 \quad \text{for} \quad x<0, \qquad f_2(x)=0 \quad \text{for} \quad x>0. \qquad (37)$$

Then,

$$\tilde{f}_1(\lambda) = \int_{-\infty}^{+\infty} dx \;\; e^{-i\lambda x}\; f_1(x) = \int_{0}^{+\infty} dx \;\; e^{-i\lambda x}\, f(x) \; = \; \tilde{f}_-(\lambda), \qquad (38a)$$

$$\tilde{f}_2(\lambda) = \int_{-\infty}^{\infty} dx \;\; e^{-i\lambda x}\; f_2(x) = \int_{-\infty}^{0} dx \;\; e^{-i\lambda x}\, f(x) \; = \; \tilde{f}_+(\lambda), \qquad (38b)$$

where $\tilde{f}_-(\lambda)$ is analytic in the ~~upper~~ lower half plane and $\tilde{f}_+(\lambda)$ is analytic in the upper half plane.

In many problems, one usually calculates the FT. $\tilde{f}(\lambda)$ of the unknown function $f(x)$ and then seeks the appropriate inversion path in order to recover $f(x)$. This step has to be taken with extreme care; the choice of the inversion path is <u>not</u> straightforward when $f(x)$ is not given a priori. For a well-posed problem, the correct path is often dictated by the "physical conditions" of the problem (for example, recall the calculation of the Green's function by knowledge of its FT and the suitable boundary or initial conditions).