Solutions of boundary-value problems by Wiener-Hopf method

① Find ψ: $\begin{cases} \psi_{xx}+\psi_{yy}+k^2\psi=0\ , & (x,y)\in R \\ \psi_y(x,0)=e^{i\alpha x}\ , & x\geq 0,\quad 0<\alpha<k\ ,\ k>0, \\ \sqrt{r}\left(\frac{\partial\psi}{\partial r}+ik\psi\right)\to 0 & \text{as}\quad r=\sqrt{x^2+y^2}\to\infty\ , \end{cases}$ [Problem 24 of Set # 8]

$R=\mathbb{R}^2-\{(x,y): y=0,\ x\geq 0\}$.
└ 2D space

Let

$$\psi(x,y)=\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\ \tilde{\psi}(\lambda,y)\ e^{i\lambda x}$$

$$\Rightarrow\quad \left(-\lambda^2+\frac{\partial^2}{\partial y^2}+k^2\right)\tilde{\psi}(\lambda,y)=0\quad\Rightarrow\quad \tilde{\psi}(\lambda,y)=\begin{cases} A(\lambda)\ e^{-\sqrt{\lambda^2-k^2}\,y}\ , & y>0, \\ B(\lambda)\ e^{\sqrt{\lambda^2-k^2}\,y}\ , & y<0\ , \end{cases}$$

with the requirement that $\sqrt{\lambda^2-k^2}\sim|\lambda|$ as $\lambda\to\pm\infty$ along the real axis so that $\tilde{\psi}(\lambda,y)$ does <u>not</u> blow up (exponentially). The question then arises of <u>how we choose the branch cuts for $\sqrt{\lambda^2-k^2}$</u>.

We essentially encountered the same question in connection with the Sommerfeld diffraction problem solved in class. In that case, we considered $k=k_1+i\epsilon$, $\epsilon>0$, and let $\epsilon\to 0^+$ ultimately. This procedure amounted to moving the branch points $\pm k$ slightly off the real axis so that $-k$ was below the real axis and $+k$ was above the real axis. The corresponding choice of branch cuts (that should <u>not</u> intersect the real axis) is the following: an infinite branch cut emanates from $-k$ and is extended to the lower λ-plane while an infinite branch cut emanates from $+k$ and is extended to the upper half plane. This choice gives $\sqrt{\zeta^2-k^2} = \sqrt{\zeta-k}\;\sqrt{\zeta+k} = -i\sqrt{k-\zeta}\;\sqrt{k+\zeta} = -i\sqrt{k^2-\zeta^2}$, $\operatorname{Im}\sqrt{\zeta^2-k^2}<0$ if $-k<\zeta<k$, $k>0$,

which in turn means that the exponential $e^{-\sqrt{\zeta^2-k^2}\,|y|} = e^{i\sqrt{k^2-\zeta^2}\,|y|}$

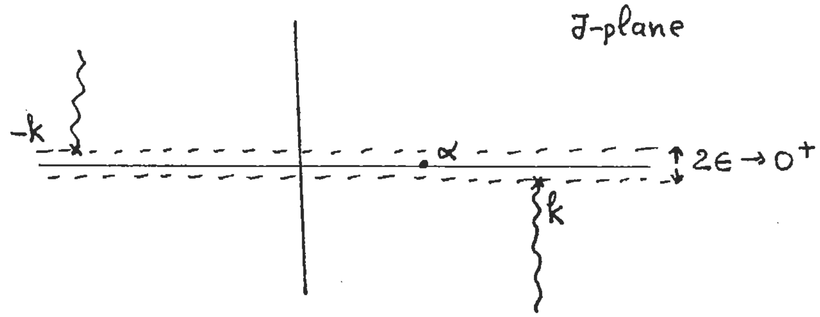
describes traveling waves of the form $e^{ip|y|}$, $p=p(\zeta)>0$. In other words, in the case of the Sommerfeld diffraction problem, setting $k=k_1+i\epsilon$, $\epsilon>0$, and restricting the inversion path on, (or close to) the real axis amounted to taking the solution as a superposition of waves of the form e^{ipy}, $p>0$. This condition is consistent with the existence of a diffracted wave of the form $\psi \sim C_1 \frac{e^{ikr}}{\sqrt{r}}$ as $r\to\infty$, where C_1 is independent of r. Note that this diffracted wave is an <u>outgoing</u> wave (traveling outward) and <u>should</u> be described as a superposition of waves of similar character, i.e., superposition of waves $\propto e^{ip|y|}$, $p>0$. With $\psi = C_1 \frac{e^{ikr}}{\sqrt{r}} + \underbrace{o\left(\frac{e^{ikr}}{\sqrt{r}}\right)}_{\to 0 \text{ faster than the leading term}}$ as $r\to\infty$, the Sommerfeld radiation condition reads

$$\sqrt{r}\left(\frac{\partial\psi}{\partial r} - ik\psi\right) \to 0 \quad \text{as} \quad r\to\infty.$$

In the present case, the Sommerfeld radiation condition is $\sqrt{r}\left(\frac{\partial\psi}{\partial r}+ik\psi\right)\to 0$, i.e., with a $+$ instead of a $-$. Thus, $\psi \sim \bar{C}_1 \frac{e^{-ikr}}{\sqrt{r}}$ as $r\to\infty$, which means that k can be replaced by $k_1 - i\epsilon$, $\epsilon>0$, or equivalently, the branch cut configuration consists of an infinite cut emanating from $-k$ and being extended to the upper ζ-plane, and an infinite cut emanating from $+k$ and being extended to the lower ζ-plane. The corresponding exponential,

$$e^{-\sqrt{\zeta^2-k^2}\,|y|} = e^{-i\sqrt{k^2-\zeta^2}\,|y|} = \lim_{\epsilon\to 0^+} e^{-i\sqrt{(k+i\epsilon)^2-\zeta^2}\,|y|}$$

describes <u>outgoing</u> traveling waves $e^{-ip|y|}$, $p=p(\zeta)>0$, if $-k<\zeta<k$ (k: real), consistent with the diffracted field $e^{-ikr}/\sqrt{r}$.



$$\psi_y(x,y) = \int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\, e^{i\zeta x}\cdot\begin{cases} -\sqrt{\zeta^2-k^2}\; A(\zeta)\; e^{-\sqrt{\zeta^2-k^2}\,y}, & y>0, \\ \sqrt{\zeta^2-k^2}\; B(\zeta)\; e^{\sqrt{\zeta^2-k^2}\,y}, & y<0. \end{cases}$$

Since $\psi_y(x,0)$ is defined unambiguously for $y=0$ and $x\geq 0$, it is reasonable to assume that

$$\psi_y(x,y) \text{ is continuous across } y=0 \text{ for all } x.$$

From the FT formula for $\psi_y(x,y)$,

$$\psi_y(x,0^\pm) = \int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\, e^{i\zeta x}\cdot\begin{cases} -\sqrt{\zeta^2-k^2}\; A(\zeta), & y=0^+, \\ \sqrt{\zeta^2-k^2}\; B(\zeta), & y=0^-. \end{cases}$$

$$\psi_y(x,0^-) = \psi_y(x,0^+) \implies -A(\zeta) = B(\zeta).$$

Since $$\psi(x,0^\pm) = \int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\, e^{i\zeta x}\begin{cases} A(\zeta), & y=0^+, \\ B(\zeta)=-A(\zeta), & y=0^-, \end{cases}$$

it follows that $\psi(x,0^-) + \psi(x,0^+) = 0.$

$\psi(x,y)$: continuous across $y=0$, $x<0$ $\Longrightarrow$ $\psi(x,0^+) = \psi(x,0^-) = 0$, $x<0.$

$$\psi(x,0^+) = \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\, e^{i\zeta x}\, A(\zeta)$$

$$\Rightarrow \quad A(\zeta) = \int_{-\infty}^{\infty} dx\, \psi(x,0^+)\, e^{-i\zeta x} = \int_{-\infty}^{0} dx\, \cancel{\psi(x,0^+)}^{\,0}\, e^{-i\zeta x} + \int_0^{\infty} dx\, \psi(x,0^+)\, e^{-i\zeta x}$$

$$= \int_0^{\infty} dx\, \psi(x,0^+)\, e^{-i\zeta x} \quad : \quad \text{"$-$ function," analytic in } \operatorname{Im}\zeta \le -\epsilon.$$

We expect this function to be analytic in $\operatorname{Im}\zeta \le -\epsilon$.

$$\psi_y(x,0) = -\int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\, e^{i\zeta x}\, \sqrt{\zeta^2-k^2}\, A(\zeta)$$

$$\Rightarrow -\sqrt{\zeta^2-k^2}\, A(\zeta) = \int_{-\infty}^{\infty} dx\, \psi_y(x,0)\, e^{-i\zeta x} = \underbrace{\int_{-\infty}^{0} dx\, \psi_y(x,0)\, e^{-i\zeta x}}_{\text{"$+$ fcn"}} + \underbrace{\int_0^{\infty} dx\, \psi_y(x,0)\, e^{-i\zeta x}}_{\text{"$-$ fcn"}}$$

By restricting ζ to $\operatorname{Im}\zeta \le -\epsilon$, the second integral is evaluated as $(\epsilon > 0)$

$$\int_0^{\infty} dx\, \psi_y(x,0)\, e^{-i\zeta x} = \int_0^{\infty} dx\, e^{i\alpha x}\, e^{-i\zeta x} = \frac{1}{i(\zeta-\alpha)}, \quad \operatorname{Im}\zeta \le -\epsilon.$$

Thus,

$$-\sqrt{\zeta^2-k^2}\, A(\zeta) = \int_{-\infty}^{0} dx\, \psi_y(x,0)\, e^{-i\zeta x} + \frac{1}{i(\zeta-\alpha)}$$

$$\Longrightarrow \quad \underset{-}{-\sqrt{\zeta^2-k^2}\, A(\zeta)} = \underset{+}{\Phi(\zeta)} + \frac{1}{i(\zeta-\alpha)}, \qquad \underset{+}{\Phi(\zeta)} \equiv \int_{-\infty}^{0} dx\, \psi_y(x,0)\, e^{-i\zeta x} \text{ ; analytic for } \operatorname{Im}\zeta > -\epsilon.$$

This is the desired relation, which holds for $\operatorname{Im}\zeta = -\epsilon$.

Note that $\sqrt{\jmath^2-k^2} = \underbrace{\sqrt{\jmath-k}}_{+} \cdot \underbrace{\sqrt{\jmath+k}}_{-}$, because k lies in the lower $\jmath$-plane. Hence,

$$-\underbrace{\sqrt{\jmath+k}\; A(\jmath)}_{-} = \underbrace{\frac{1}{\sqrt{\jmath-k}}}_{+} \underbrace{\Phi(\jmath)}_{+} + \underbrace{\frac{1}{\sqrt{\jmath-k}}}_{+} \underbrace{\frac{1}{i(\jmath-\alpha)}}_{-},$$

where

$$\frac{1}{\sqrt{\jmath-k}} \frac{1}{i(\jmath-\alpha)} = \underbrace{\frac{1}{i(\jmath-\alpha)}\left(\frac{1}{\sqrt{\jmath-k}} - \frac{1}{\sqrt{\alpha-k}}\right)}_{+} + \underbrace{\frac{1}{i(\jmath-\alpha)} \frac{1}{\sqrt{\alpha-k}}}_{-},$$

leading to

$$-\underbrace{\sqrt{\jmath+k}\; A(\jmath)}_{-} - \underbrace{\frac{1}{i(\jmath-\alpha)} \frac{1}{\sqrt{\alpha-k}}}_{-} = \underbrace{\frac{1}{\sqrt{\jmath-k}}}_{+} \underbrace{\Phi(\jmath)}_{+} + \underbrace{\frac{1}{i(\jmath-\alpha)}\left(\frac{1}{\sqrt{\jmath-k}} - \frac{1}{\sqrt{\alpha-k}}\right)}_{+}$$

$$\equiv E(\jmath) : \text{entire}.$$

Because $\frac{1}{\sqrt{\jmath-k}} \Phi(\jmath) + \frac{1}{i(\jmath-\alpha)}\left(\frac{1}{\sqrt{\jmath-k}} - \frac{1}{\sqrt{\alpha-k}}\right) \to 0$ as $|\jmath| \to \infty$,

it follows that

$$E(\jmath) \equiv 0.$$

Hence,

$$A(\jmath) = + \frac{i}{\sqrt{\alpha-k}} \frac{1}{\jmath-\alpha} \frac{1}{\sqrt{\jmath+k}}$$

$$\Rightarrow \quad \psi(x, 0^+) = \int_C \frac{d\jmath}{2\pi} e^{i\jmath x} \frac{i}{\sqrt{\alpha-k}} \frac{1}{\jmath-\alpha} \frac{1}{\sqrt{\jmath+k}} = \int_C \frac{d\jmath}{2\pi} \frac{e^{i\jmath x}}{\sqrt{k-\alpha}} \frac{1}{\jmath-\alpha} \frac{1}{\sqrt{\jmath+k}}$$

where C lies slightly below the real axis and $\sqrt{\alpha-k} = i\sqrt{k-\alpha}$.

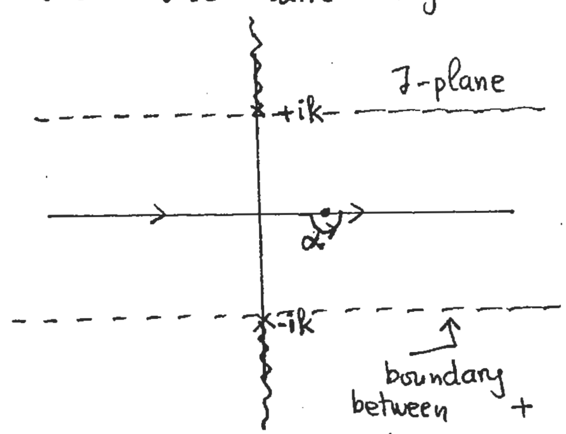
② Find ψ: $\begin{cases} \psi_{xx} + \psi_{yy} - k^2\psi = 0, & (x,y) \in R, \\ \psi_y(x,0) = e^{i\alpha x}, & x > 0, \\ \psi \to 0 \text{ as } & r \to \infty. \end{cases}$ $\quad R = \underbrace{\mathbb{R}^2}_{\text{2D space}} - \{(x,y):\ x \geq 0,\ y = 0\}$,

Set

$$\psi(x,y) = \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\ \tilde{\psi}(\zeta,y)\ e^{i\zeta x}$$

$$\Rightarrow \left(-\zeta^2 + \frac{\partial^2}{\partial y^2} - k^2\right)\tilde{\psi}(\zeta,y) = 0 \quad \Rightarrow \quad \tilde{\psi}(\zeta,y) = \begin{cases} A(\zeta)\ e^{-\sqrt{\zeta^2+k^2}\,y}, & y > 0, \\ B(\zeta)\ e^{\sqrt{\zeta^2+k^2}\,y}, & y < 0. \end{cases}$$

Notice that the branch points are now $\pm ik$, i.e., they are now located on the <u>imaginary</u> axis. Hence, there is no need to introduce an $\epsilon > 0$ by applying a radiation condition. Accordingly, the first Riemann sheet for $\sqrt{\zeta^2+k^2}$ is defined so as to render the Fourier integral convergent. Consequently, $\psi \to 0$ as $r \to \infty$, by taking the inversion path on the real axis, indented below $\zeta = +\alpha$.



This problem can be thought of as stemming from the analytic continuation of the quantities in Prob. 5 under

$$k \Rightarrow -ik, \quad k > 0.$$

(Accordingly, the diffracted field $\frac{e^{ikr}}{\sqrt{r}} \Longrightarrow \frac{e^{-kr}}{\sqrt{r}} \to 0$ as $r \to \infty$.)

$$\psi(x,y) = \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\, e^{i\zeta x} \begin{cases} A(\zeta)\ e^{-\sqrt{\zeta^2+k^2}\,y}, & y > 0, \\ B(\zeta)\ e^{\sqrt{\zeta^2+k^2}\,y}, & y < 0. \end{cases}$$

$$\Rightarrow \quad \psi_y(x,y) = \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\, e^{i\zeta x} \begin{cases} -\sqrt{\zeta^2+k^2}\ A(\zeta)\ e^{-\sqrt{\zeta^2+k^2}\,y}, & y > 0, \\ \sqrt{\zeta^2+k^2}\ B(\zeta)\ e^{\sqrt{\zeta^2+k^2}\,y}, & y < 0. \end{cases}$$

Similarly to Prob. 5, we infer that

$$A(\zeta) = -B(\zeta),$$

$$\psi(x,0^+) + \psi(x,0^-) = 0 \quad \underset{\text{for } y=0,\ x<0}{\overset{\psi:\text{continuous}}{\Longrightarrow}} \quad \psi(x,0^+) = \psi(x,0^-) = 0, \qquad x<0.$$

$$A(\zeta) = \int_{-\infty}^{\infty} dx \ \psi(x,0^+)\, e^{-i\zeta x} = \int_{0}^{\infty} dx \ \psi(x,0^+)\, e^{-i\zeta x} : \quad \text{"}-\text{ function."}$$

By taking $\psi(x,y)$ to be <u>integrable</u> in x, $A(\zeta)$ must be <u>analytic</u> in $\mathrm{Im}\,\zeta < 0$.
Integrability is achieved by taking $\alpha = \alpha_1 + i\delta$, $\delta > 0$, $\delta \to 0^+$.

$$-\underset{-}{\sqrt{\zeta^2+k^2}}\, A(\zeta) = \int_{-\infty}^{\infty} dx \ \psi_y(x,0^+)\, e^{-i\zeta x} = \int_{-\infty}^{0} dx \ \psi_y(x,0^+)\, e^{-i\zeta x} + \int_{0}^{\infty} dx \ \psi_y(x,0^+)\, e^{-i\zeta x}$$

$$= \underset{+}{\Phi(\zeta)} + \underset{-}{\frac{1}{i(\zeta-\alpha)}}, \qquad \Phi(\zeta) \equiv \int_{-\infty}^{0} dx \ \psi_y(x,0^+)\, e^{-i\zeta x} : \ \text{"}+\text{ function,"}$$

analytic for $\mathrm{Im}\,\zeta > 0$.

$$\Rightarrow \ -\underset{+}{\sqrt{\zeta+ik}}\ \underset{-}{\sqrt{\zeta-ik}}\ \underset{-}{A(\zeta)} = \underset{+}{\Phi(\zeta)} + \underset{-}{\frac{1}{i(\zeta-\alpha)}}, \qquad \zeta:\text{real}.$$

$$\Rightarrow \ \underset{-}{\sqrt{\zeta-ik}\ A(\zeta)} + \underset{-}{\frac{1}{\sqrt{\alpha+ik}}\ \frac{1}{i(\zeta-\alpha)}} = \underset{-}{} \frac{-1}{\underset{+}{\sqrt{\zeta+ik}}}\ \underset{+}{\Phi(\zeta)} + \underbrace{\frac{i}{\zeta-\alpha}\left(\frac{1}{\sqrt{\zeta+ik}} - \frac{1}{\sqrt{\alpha+ik}}\right)}_{+}$$

$$\equiv E(\zeta).$$

It follows that $\quad E(\zeta) \equiv 0.$

Hence,

$$A(\zeta) = \frac{+i}{\sqrt{\alpha+ik}}\ \frac{1}{\sqrt{\zeta-ik}}\ \frac{1}{\zeta-\alpha}$$

$$\Rightarrow \ \psi(x,0^+) = \int_C \frac{d\zeta}{2\pi}\, e^{i\zeta x}\ \frac{i}{\sqrt{\alpha+ik}}\ \frac{1}{\sqrt{\zeta-ik}}\ \frac{1}{\zeta-\alpha}, \quad \text{where } C \text{ lies below } \zeta=\alpha.$$

③ Find ϕ: $\begin{cases} \phi_{xx} + \phi_{yy} + k^2\phi = 0, & (x,y) \in R, \quad R = \mathbb{R}^2 - \{(x,y): y=0, x>0\} \\ \phi_x(x,0) = 0, & x>0, \\ \phi_y : \text{continuous across } y=0 & \text{for } x<0, \\ \phi_x : \text{continuous across } y=0 & \text{for } \underline{\text{all}} \ x, \end{cases}$

where $$\phi(x,y) = \underbrace{e^{-ikx\cos\theta - iky\sin\theta}}_{\text{incident field}} + \underbrace{\psi(x,y)}_{\text{scattered field}}, \qquad \pi/2 < \theta < \pi.$$

$$\Rightarrow \phi_x(x,y) = -ik\cos\theta \; e^{-ikx\cos\theta - iky\sin\theta} + \psi_x(x,y).$$

We formulate the problem in terms of the scattered field, $\psi(x,y)$:

$$\begin{cases} \psi_{xx} + \psi_{yy} + k^2\psi = 0, & (x,y)\in R, \\ \psi_x(x,0) = ik\cos\theta \; e^{-ikx\cos\theta}, & x>0, \\ \psi_y : \text{continuous across } y=0 \text{ for } x<0, \\ \psi_x : \text{continuous across } y=0 \text{ for } \underline{\text{all}} \ x. \\ \psi, \psi_y : \underset{\text{in } x}{\text{integrable}} \text{ with } k = k_1 + i\epsilon, \ \epsilon>0, \end{cases}$$

as was shown in class. Define

$$\psi(x,y) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \, e^{i\lambda x} \, \tilde{\psi}(\lambda, y)$$

$$\Rightarrow \quad \tilde{\psi}(\lambda, y) = \begin{cases} A(\lambda) \; e^{-\sqrt{\lambda^2-k^2}\,y}, & y>0, \\ B(\lambda) \; e^{\sqrt{\lambda^2-k^2}\,y}, & y<0. \end{cases}$$

Hence,

$$\psi(x,y) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \, e^{i\lambda x} \begin{cases} A(\lambda) \; e^{-\sqrt{\lambda^2-k^2}\,y}, & y>0, \\ B(\lambda) \; e^{\sqrt{\lambda^2-k^2}\,y}, & y<0, \end{cases}$$

$$\Rightarrow \quad \psi_x(x,y) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \, i\lambda \, e^{i\lambda x} \cdot \begin{cases} A(\lambda) \; e^{-\sqrt{\lambda^2-k^2}\,y}, & y>0, \\ B(\lambda) \; e^{\sqrt{\lambda^2-k^2}\,y}, & y<0. \end{cases}$$

$$\psi_x(x, y=0^{\pm}) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \, i\lambda \, e^{i\lambda x} \cdot \begin{cases} A(\lambda), & y=0^+ \\ B(\lambda), & y=0^-. \end{cases}$$

$$\psi_x(x,y=0^+) = \psi_x(x,y=0^-) \overset{\text{all } x}{\Rightarrow} \quad \underline{A(\zeta) = B(\zeta)}.$$

$$\Rightarrow \quad \psi(x,0^+) = \psi(x,0^-) \quad \text{for } \underline{\text{all}} \quad x.$$

$$\psi_y(x,y) = \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} e^{i\zeta x} \begin{cases} -\sqrt{\zeta^2-k^2}\, A(\zeta)\; e^{-\sqrt{\zeta^2-k^2}\, y}, & y>0, \\ \sqrt{\zeta^2-k^2}\; B(\zeta)\; e^{\sqrt{\zeta^2-k^2}\, y}, & y<0 \end{cases}$$

$$\Rightarrow \quad \psi_y(x,0^\pm) = \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} e^{i\zeta x} \cdot \begin{cases} -\sqrt{\zeta^2-k^2}\, A(\zeta), & y>0, \\ \sqrt{\zeta^2-k^2}\; B(\zeta), & y<0. \end{cases} \qquad (A \equiv B).$$

It follows that

$$[A(\zeta)=B(\zeta) \Rightarrow] \qquad \psi_y(x,0^+) + \psi_y(x,0^-) = 0, \qquad \underline{\text{all}} \quad x.$$

But $\quad \psi_y(x,0^+) = \psi_y(x,0^-) \quad$ for $\quad x<0$

$$\Rightarrow \quad \psi_y(x,0^+) = \psi_y(x,0^-) = 0 \text{ for } \quad x<0.$$

$$-\sqrt{\zeta^2-k^2}\; A(\zeta) = \int_{-\infty}^{\infty} dx \;\; \psi_y(x,0^+)\; e^{-i\zeta x} = \int_0^{\infty} dx \;\; \psi_y(x,0^+)\; e^{-i\zeta x} \; : \text{ "}-\text{ function."}$$

$$\equiv \quad F_-(\zeta) \quad : \text{ analytic in } \operatorname{Im}\zeta \le -\epsilon.$$

$$i\zeta\, A(\zeta) = \int_{-\infty}^{\infty} dx \;\; \psi_x(x,y=0)\; e^{-i\zeta x} = \underbrace{\int_{-\infty}^{0} dx \;\; \psi_x(x,0)\, e^{-i\zeta x}}_{+} + \underbrace{\int_0^{\infty} dx \;\; \psi_x(x,0) \;\; e^{-i\zeta x}}_{-}$$

$$= \Phi_+(\zeta) + \frac{ik\cos\theta}{i(\zeta+k\cos\theta)}, \qquad \Phi_+(\zeta) \equiv \int_{-\infty}^{0} dx \;\; \psi_x(x,0)\; e^{-i\zeta x} \text{ : analytic for } \operatorname{Im}\zeta > -\epsilon.$$

Thus, by elimination of $A(\zeta)$,

$$i\zeta \; \frac{-F_-(\zeta)}{\sqrt{\zeta^2-k^2}} = \Phi_+(\zeta) + \frac{k\cos\theta}{\zeta+k\cos\theta}, \qquad \operatorname{Im}\zeta = -\epsilon,$$

$$\Rightarrow \quad \underset{-}{-\frac{i\zeta}{\sqrt{\zeta-k}}}\, F_-(\zeta) = \underset{+}{\sqrt{\zeta+k}\;\Phi_+(\zeta)} + k\cos\theta \; \underset{+}{\sqrt{\zeta+k}}\; \underset{-}{\frac{1}{\zeta+k\cos\theta}} ,$$

where
$$\frac{\sqrt{\zeta+k}}{\zeta+k\cos\theta} = \underbrace{\frac{1}{\zeta+k\cos\theta}\left(\sqrt{\zeta+k} - \sqrt{k-k\cos\theta}\right)}_{+} + \underbrace{\frac{\sqrt{k-k\cos\theta}}{\zeta+k\cos\theta}}_{-} .$$

$$\Rightarrow \quad \underset{-}{\frac{-i\zeta}{\sqrt{\zeta-k}}}\, F_-(\zeta) - \underset{-}{k\cos\theta\,\frac{\sqrt{k(1-\cos\theta)}}{\zeta+k\cos\theta}} = \underset{+}{\sqrt{\zeta+k}\;\Phi_+(\zeta)} + \underbrace{\frac{k\cos\theta}{\zeta+k\cos\theta}\left(\sqrt{\zeta+k} - \sqrt{k(1-\cos\theta)}\right)}_{+} \equiv \underset{\text{entire}}{E(\zeta)} .$$

From the fact that $\dfrac{-i\zeta}{\sqrt{\zeta-k}} F_-(\zeta) - k\cos\theta \cdot \dfrac{\sqrt{k(1-\cos\theta)}}{\zeta+k\cos\theta} \longrightarrow 0$ as $|\zeta| \to \infty$,

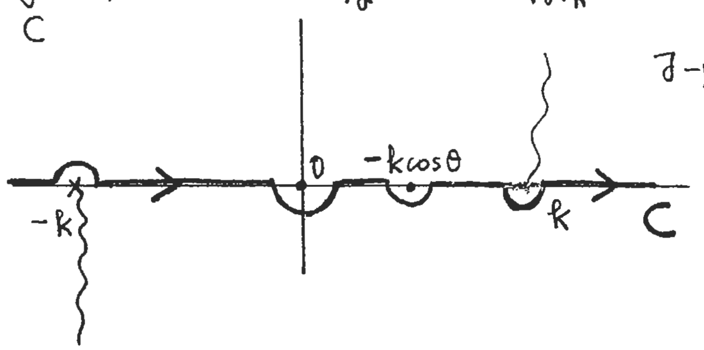
it follows that
$$E(\zeta) \equiv 0.$$

Hence,
$$F_-(\zeta) = \frac{\sqrt{\zeta-k}}{-i\zeta}\; k\cos\theta \; \frac{\sqrt{k(1-\cos\theta)}}{\zeta+k\cos\theta} .$$

Accordingly,
$$A(\zeta) = -\frac{F_-(\zeta)}{\sqrt{\zeta^2-k^2}} = k\cos\theta\,\sqrt{k(1-\cos\theta)}\;\frac{1}{i\zeta}\;\frac{1}{\sqrt{\zeta+k}}\;\frac{1}{\zeta+k\cos\theta}$$

$$\Rightarrow \quad \psi(x,y) = \int_C \frac{d\zeta}{2\pi}\, e^{i\zeta x}\, \frac{k\cos\theta\,\sqrt{k(1-\cos\theta)}}{i\zeta}\, \frac{1}{\sqrt{\zeta+k}}\, \frac{1}{\zeta+k\cos\theta}\, e^{-\sqrt{\zeta^2-k^2}\,|y|} .$$

ζ-plane $\quad (\epsilon \to 0^+)$.



Then, $\phi(x,y)$ follows by
$$\phi = e^{-ikx\cos\theta - iky\sin\theta} + \psi$$