Erratum for Solutions to Set 7

Definition of error function: $\mathrm{erf}(z) \equiv \frac{2}{\sqrt{\pi}}\int_0^z dt\, e^{-t^2} = \frac{1}{\sqrt{\pi}}\int_0^{z^2} dy\, \frac{e^{-y}}{\sqrt{y}}$

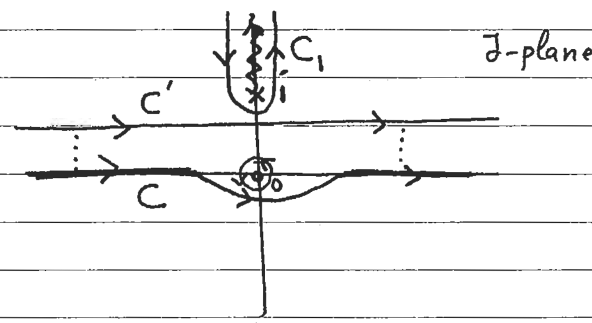
Correction to solution of Prob. 22 starting from p. 5 of Set 7.

(22) On p.5 of the Solution Set 7, I give the formula

$$u(x) = e^{i\pi/4}\int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\, e^{i\zeta x}\,\frac{\sqrt{\zeta - i}}{i\zeta}, \qquad x>0,$$

where the path of integration is deformed below the pole at $\zeta=0$.

Correct procedure to find $u(x)$:



We shift the path above the pole at $\zeta=0$, by picking up the residue:

$$u(x) = e^{i\pi/4}\int_{C'}\frac{d\zeta}{2\pi}\, e^{i\zeta x}\,\frac{\sqrt{\zeta-i}}{i\zeta} + 2\pi i\,\frac{e^{i\pi/4}}{2\pi i}\,\underset{\zeta=0}{\mathrm{Res}}\left[e^{i\zeta x}\,\frac{\sqrt{\zeta-i}}{\zeta}\right]$$

$$= e^{i\pi/4}\int_{C_1}\frac{d\zeta}{2\pi i}\, e^{i\zeta x}\,\frac{\sqrt{\zeta-i}}{\zeta} + e^{i\pi/4}\cdot\sqrt{\zeta-i}\,\Big|_{\zeta=0}, \qquad x>0$$

where C_1 is the path wrapped around the branch cut emanating from $\zeta=i$, and $\sqrt{\zeta-i}\,\Big|_{\zeta=0} = e^{-i\pi/4}$. So,

$$u(x) = e^{i\pi/4}\int_{C_1}\frac{d\zeta}{2\pi i}\, e^{i\zeta x}\,\frac{\sqrt{\zeta-i}}{\zeta} + 1, \qquad x>0.$$

We integrate along the right and left sides of the branch cut.

$$\zeta - i = e^{i\pi/2} \cdot y$$

Right side, $\zeta = i + iy$: $\sqrt{\zeta - i} = \sqrt{iy} = e^{i\pi/4} \cdot \sqrt{y}$, $\sqrt{y} > 0$ if $y > 0$.

Left side, $\zeta = i + iy$: $\sqrt{\zeta - i} = -e^{i\pi/4} \cdot \sqrt{y}$, $\sqrt{y} > 0$.

$$\zeta - i = e^{i(\frac{\pi}{2} - 2\pi)} \cdot y \qquad\qquad e^{i\zeta x} = e^{-x} \cdot e^{-xy} \text{ in either side.}$$

So,

$$e^{i\pi/4} \int_{C_1} \frac{d\zeta}{2\pi i} e^{i\zeta x} \frac{\sqrt{\zeta - i}}{\zeta} = e^{i\pi/4} \left\{ \int_0^\infty \frac{d(iy)}{2\pi i} e^{-x} e^{-xy} \frac{e^{i\pi/4}\sqrt{y}}{i(1+y)} - \int_0^\infty \frac{d(iy)}{2\pi i} e^{-x} e^{-xy} \frac{(-e^{i\pi/4}\sqrt{y})}{i(1+y)} \right\}$$

$$= 2 \cdot \frac{e^{-x}}{2\pi} \int_0^\infty dy \cdot e^{-xy} \frac{\sqrt{y}}{1+y} = \frac{e^{-x}}{\pi} \int_0^\infty dy \, e^{-xy} \frac{\sqrt{y}}{1+y}, \quad x > 0.$$

It follows that

$$u(x) = \frac{e^{-x}}{\pi} \int_0^\infty dy \, e^{-xy} \frac{\sqrt{y}}{1+y} + 1, \quad \underline{x > 0}.$$

Notice that for $x \to +\infty$, the integral in the RHS approaches 0. Hence,

$$u(x) \longrightarrow 1 \quad \text{as} \quad x \to +\infty \quad !$$

This is a nice result. The procedure that we applied (Wiener-Hopf method of factorization by FTs) <u>also</u> covers for cases where $u(x) \to$ const. as $x \to \infty$, const. $\neq 0$.

It follows that

$$\frac{\partial u}{\partial x} = \frac{-1}{\pi} \int_0^\infty dy \, e^{-x(1+y)} \sqrt{y} = -\frac{1}{\pi} e^{-x} \cdot \frac{1}{x^{3/2}} \cdot \Gamma(3/2) = -\frac{1}{2\sqrt{\pi}} \frac{e^{-x}}{x^{3/2}}.$$

same as in Set i

Hence, since $u \to 1$ as $x \to \infty$,

$$u(x) = 1 - \int_x^\infty dx' \frac{\partial u}{\partial x'} = 1 + \frac{1}{2\sqrt{\pi}} \int_x^\infty dx' \frac{e^{-x'}}{x'^{3/2}} = 1 - \frac{1}{\sqrt{\pi}} \int_x^\infty d(x'^{-1/2}) \, e^{-x'}$$

$$= 1 + \frac{e^{-x}}{\sqrt{\pi x}} - \frac{1}{\sqrt{\pi}} \int_x^\infty dx' \frac{e^{-x'}}{\sqrt{x'}} = \frac{e^{-x}}{\sqrt{\pi x}} + \frac{1}{\sqrt{\pi}} \int_0^x dx' \frac{e^{-x'}}{\sqrt{x'}} = \frac{e^{-x}}{\sqrt{\pi x}} + \mathrm{erf}(\sqrt{x}), \; x > 0.$$