18.307: Integral Equations M.I.T. Department of Mathematics

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Solutions to Homework 6

(18.) $$u(x) = \lambda \int_{-\infty}^{\infty} dy \; e^{-ixy} \; u(y) \; , \quad -\infty < x < +\infty.$$

(a) We write this equation symbolically as

$$u(x) = \lambda \, \underset{\substack{\text{integral}\\ \text{operator}}}{\mathcal{F}} u(x) \qquad \text{where} \quad \mathcal{F} u \equiv \int_{-\infty}^{\infty} dy \; e^{-ixy} \; u(y)$$

It follows that

$$\mathcal{F}^2 u(x) = \mathcal{F} \cdot \mathcal{F} u(x) = 2\pi \cdot u(-x)$$

Then,

$$\left.\begin{aligned} &\mathcal{F}^4 u(x) = \mathcal{F}^2 \mathcal{F}^2 u(x) = (2\pi)^2 \, u(x) \\ &\frac{1}{\lambda} u(x) = \mathcal{F} u(x) \Rightarrow \mathcal{F}^4 u(x) = \frac{1}{\lambda^4} \, u(x) \end{aligned}\right\} \Rightarrow \frac{1}{\lambda^4} u(x) = (2\pi)^2 \cdot u(x) \quad u(x) \neq 0$$

$$\Rightarrow \lambda^4 = \frac{1}{(2\pi)^2} \Rightarrow \lambda = \frac{\pm 1}{\sqrt{2\pi}} \, , \; \frac{\pm i}{\sqrt{2\pi}}$$

(b) Consider $u_n(x) = e^{-x^2/2} \, H_n(x)$, $H_n(x) = (-1)^n \, e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

The FT of $u_n(x)$ is

$$\tilde{u}_n(k) = \int_{-\infty}^{\infty} dx \; e^{-ikx} \; e^{-x^2/2} \, H_n(x) = \int_{-\infty}^{\infty} dx \; e^{-ikx} \; e^{-x^2/2} \, (-1)^n \, e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$\Rightarrow \quad \tilde{u}_n(k) = (-1)^n \int_{-\infty}^{+\infty} dx \quad e^{-ikx} \, e^{x^2/2} \, \frac{d^n}{dx^n} \, e^{-x^2} .$$

We move the differential operators to the right as follows.

$$e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2} = e^{x^2/2} \frac{d^n}{dx^n} \left(e^{-x^2/2} e^{-x^2/2} \right) = e^{x^2/2} \frac{d^{n-1}}{dx^{n-1}} \frac{d}{dx} \left(e^{-x^2/2} e^{-x^2/2} \right)$$

$$= e^{x^2/2} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2/2} \left(\frac{d}{dx} - x \right) e^{-x^2/2} = e^{x^2/2} \frac{d^{n-2}}{dx^{n-2}} \frac{d}{dx} e^{-x^2/2} \left(\frac{d}{dx} - x \right) e^{-x^2/2}$$

$$= e^{x^2/2} \frac{d^{n-2}}{dx^{n-2}} e^{-x^2/2} \left(\frac{d}{dx} - x \right) \left(\frac{d}{dx} - x \right) e^{-x^2/2}$$

$$= \cdots = e^{x^2/2} e^{-x^2/2} \underbrace{\left(\frac{d}{dx} - x \right) \left(\frac{d}{dx} - x \right) \cdots \left(\frac{d}{dx} - x \right)}_{n \text{ factors}} e^{-x^2/2}$$

$$= \left(\frac{d}{dx} - x \right)^n e^{-x^2/2} ,$$

by repeated use of the identity $\quad \frac{d}{dx} e^{g(x)} f(x) = e^{g(x)} \left[\frac{d}{dx} + g'(x) \right] f(x).$

It follows that

$$\tilde{u}_n(k) = (-1)^n \int_{-\infty}^{\infty} dx \quad e^{-ikx} \left(\frac{d}{dx} - x \right)^n e^{-x^2/2} .$$

Integration by parts gives

$$\int_{-\infty}^{\infty} dx \; f(x) \left(\frac{d}{dx} - x \right) g(x) = \cancelto{0}{f(x) g(x) \Big|_{x=-\infty}^{+\infty}}$$

$$+ \int_{-\infty}^{\infty} dx \; g(x) \left(-\frac{d}{dx} - x \right) f(x) = \int_{-\infty}^{\infty} dx \; g(x) \left(-\frac{d}{dx} - x \right) f(x) ,$$

assuming that $\lim_{|x| \to +\infty} [f(x) g(x)] = 0.$

Hence, we get

$$\tilde{u}_n(k) = (-1)^n \int_{-\infty}^{\infty} dx \; e^{-x^2/2} \left(-\frac{d}{dx}-x\right)\left(-\frac{d}{dx}-x\right)\cdots\left(\frac{-d}{dx}-x\right) e^{-ikx},$$

where $\left(-\frac{d}{dx}-x\right) e^{-ikx} = \left(ik - i\frac{d}{dk}\right) e^{-ikx} = i\left(k - \frac{d}{dk}\right) e^{-ikx}$.

We thus obtain

$$\tilde{u}_n(k) = (-1)^n \, i^n \int_{-\infty}^{+\infty} dx \; e^{-x^2/2} \left(k-\frac{d}{dk}\right)\left(k-\frac{d}{dk}\right)\cdots\left(k-\frac{d}{dk}\right) e^{-ikx}$$

$$= (-i)^n \left(k-\frac{d}{dk}\right)\cdots\left(k-\frac{d}{dk}\right) \underbrace{\int_{-\infty}^{\infty} dx \; e^{-x^2/2} \; e^{-ikx}}$$

$$= i^n \left(\frac{d}{dk}-k\right)\cdots\left(\frac{d}{dk}-k\right) \cdot \sqrt{2\pi}\; e^{-k^2/2} = i^n \sqrt{2\pi}\; e^{k^2/2} \frac{d^n}{dk^n} e^{-k^2}$$

$$= (-i)^n \sqrt{2\pi}\; (-1)^n e^{k^2/2} \frac{d^n}{dk^n} e^{-k^2} = \sqrt{2\pi}\,(-i)^n \, u_n(k)$$

$$\Rightarrow \quad \tilde{u}_n(k) = \sqrt{2\pi}\,(-i)^n \, u_n(k).$$

The given equation for $u(x) = u_n(x)$ gives

$$u_n(x) = \lambda\, \tilde{u}_n(x) = \lambda \sqrt{2\pi}\,(-i)^n u_n(x) \Rightarrow \lambda_n = \frac{i^n}{\sqrt{2\pi}}$$

For $n = 0, 1, 2, 3$, $\lambda = \frac{1}{\sqrt{2\pi}}, \frac{i}{\sqrt{2\pi}}, \frac{-1}{\sqrt{2\pi}}, \frac{-i}{\sqrt{2\pi}}$.

Clearly, the eigenvalues are infinitely degenerate.

The $u_n(x) = e^{-x^2/2} H_n(x)$ are solutions of the eigenvalue problem of the quantum harmonic oscillator:

$$\left(-\frac{d^2}{dx^2}+x^2\right) u_n(x) = E_n u_n(x) \quad , \qquad u_n(|x|\to+\infty)=0.$$

Notice that the F.T. of both sides of this equation gives the same ode:

$$\left(k^2-\frac{d^2}{dk^2}\right)\tilde{u}_n(k) = E_n \tilde{u}_n(k) \, , \qquad \tilde{u}_n(|k|\to+\infty)=0,$$

i.e., one expects that $\tilde{u}_n(x)$ and $u_n(x)$ are linearly dependent.

(b) Suppose that $u(x)$ is a solution of the given integral equation. There are four possibilities:

(i) $u(x)$ corresponds to eigenvalue $\lambda_0=\frac{1}{\sqrt{2\pi}}$, and hence is a linear combination of $u_n(x)$ with $n=0,4,8,\ldots = 4m$ $(m=0,1,\ldots)$, with arbitrary coefficients.

$$u(x)=\sum_{\substack{n=0\\ n=0,4,\ldots\\ n=4m}}^{\infty} \overset{\text{arbitrary}}{\overset{\downarrow}{c_n}}\, u_n(x) = \sum_{\substack{n=0\\ n=2m}}^{\infty}\frac{1}{2}c_n\left[1+(-i)^n\right]u_n(x)\,,$$ by extending n

to all even non-negative integers and taking arbitrary (admissible) coefficients $c_2, c_6, \ldots, c_{2m}, \ldots$.

[Since $u(x)$ has to be square integrable, we need $\sum_{m=0}^{\infty}|c_{4m}|^2<\infty$; similarly $\sum_{m=0}^{\infty}|c_{2m}|^2<\infty$.]

We thus get
$$u(x)=\sum_{\substack{n=0\\ n=2m}}^{\infty}\frac{1}{2}c_n u_n(x) + \sum_{\substack{n=0\\ n=2m}}^{\infty}\frac{c_n}{2}(-i)^n u_n(x)$$

$$=\sum_{n:\text{even}}\frac{c_n}{2}u_n(x) + \sum_{n:\text{even}}\frac{c_n}{2\sqrt{2\pi}}\tilde{u}_n(x)$$

$$\rightarrow \quad u(x) = \sum_{n:\text{even}} \frac{c_n}{2} u_n(x) + \frac{1}{\sqrt{2\pi}} \, FT\left\{ \sum_{n:\text{even}} \frac{c_n}{2} u_n(x) \right\}$$

$$= f(x) + \frac{1}{\sqrt{2\pi}} \tilde{f}(x),$$

where $f(x)$: arbitrary <u>even</u> sq. integrable function (hence, $\tilde{f}(x)$ is also even and $u(x)$: even)

Thus, $u(x) = f(x) + C_1 \tilde{f}(x)$, $\quad C_1 = \frac{1}{\sqrt{2\pi}}$, $\quad \lambda = \lambda_0 = \frac{1}{\sqrt{2\pi}}$.

(ii) $\lambda = \lambda_1 = \frac{i}{\sqrt{2\pi}}$, $\quad n = 1, 5, 9, \ldots = 4m+1$, $\quad m = 0, 1, \ldots$

$$u(x) = \sum_{\substack{n=0 \\ n=4m+1}}^{\infty} c_n u_n(x) = \sum_{\substack{n=0 \\ n:\text{odd}}}^{\infty} \frac{c_n}{2} \left[1 + i(-i)^n \right] u_n(x) = \sum_{n:\text{odd}} \frac{c_n}{2} u_n(x) + i \sum_{n:\text{odd}} \frac{c_n}{2} (-i)^n u_n(x)$$

$$= \sum_{n:\text{odd}} \frac{c_n}{2} u_n(x) + \frac{i}{\sqrt{2\pi}} \sum_{n:\text{odd}} \frac{c_n}{2} \tilde{u}_n(x) = f(x) + \frac{i}{\sqrt{2\pi}} \tilde{f}(x),$$

where $f(x)$ is any arbitrary <u>odd</u> square integrable function.

(iii) $\lambda = \lambda_2 = \frac{-1}{\sqrt{2\pi}}$, $\quad n = 2, 6, 10, \ldots = 4m+2$, $\quad m = 0, 1, \ldots$

$$u(x) = \sum_{\substack{n=0 \\ n=4m+2}}^{\infty} c_n u_n(x) = \sum_{\substack{n=0 \\ n:\text{even}}}^{\infty} \frac{c_n}{2} \left[1 - (-i)^n \right] u_n(x) = \sum_{n:\text{even}} \frac{c_n}{2} u_n(x) - \sum_{n:\text{even}} \frac{c_n}{2} (-i)^n u_n(x)$$

$$= f(x) - \frac{1}{\sqrt{2\pi}} \tilde{f}(x), \qquad f(x) : \underline{\text{even}} \text{ sq. integrable function}$$

(iv) $\lambda = \lambda_3 = \dfrac{-i}{\sqrt{2\pi}}$, $n = 3, 7, 11, \ldots = 4m+3$, $m = 0, 1, 2, \ldots$

$$u(x) = \sum_{\substack{n=0 \\ n=4m+3}}^{\infty} c_n u_n(x) = \sum_{\substack{n=0 \\ n:\text{odd}}}^{\infty} \frac{c_n}{2} \left[1 - i(-i)^n\right] u_n(x) = f(x) - \frac{i}{\sqrt{2\pi}} \tilde{f}(x),$$

$f(x)$: <u>odd</u> sq. integrable function.

In summary, $u(x) = f(x) + C \cdot \tilde{f}(x)$ where $C = \begin{cases} \dfrac{1}{\sqrt{2\pi}}, & \text{for } \lambda = \dfrac{1}{\sqrt{2\pi}} \\ -1/\sqrt{2\pi}, & \lambda = -1/\sqrt{2\pi} \\ \dfrac{\pm i}{\sqrt{2\pi}}, & \lambda = \dfrac{\pm i}{\sqrt{2\pi}}. \end{cases}$

(d) Let $f(x) = e^{-ax^2/2}$ (f : even).

The F.T. of $f(x)$ is $\tilde{f}(k) = \displaystyle\int_{-\infty}^{+\infty} dx\, e^{-ikx} e^{-ax^2/2} \overset{\sqrt{a}x = y}{=\!=} \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} dy\, e^{-i\frac{k}{\sqrt{a}}y} e^{-y^2/2}$

$$= \frac{1}{\sqrt{a}} \cdot \sqrt{2\pi}\; e^{-\left(\frac{k}{\sqrt{a}}\right)^2 \cdot \frac{1}{2}} = \sqrt{\frac{2\pi}{a}}\; e^{-k^2/2a}$$

The desired solution $u(x)$ is

$$u(x) = f(x) \pm \frac{1}{\sqrt{2\pi}} \tilde{f}(x) = e^{-ax^2/2} \pm \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2\pi}{a}}\; e^{-x^2/2a}$$

$$= e^{-ax^2/2} \pm \frac{1}{\sqrt{a}}\, e^{-x^2/2a},$$

which satisfies the given equation with eigenvalue $\lambda = \dfrac{\pm 1}{\sqrt{2\pi}}$ $(n=0, 2)$.

Note that for <u>$a=1$</u>, one of the two solutions becomes $\equiv 0$, while the other one gives the Gaussian as solution, with (sole) eigenvalue $\lambda = \dfrac{1}{\sqrt{2\pi}}$.

In the following problem (Prob. 19) the kernel has a discontinuity <u>across</u> the line $y=x$. For such kernels the eigenfunctions $u_n(x)$ are continuous, i.e., such discontinuities in the kernel do <u>not</u> affect similarly the analytic properties of their eigenfunctions.

In contrast, if the kernel had a discontinuities at <u>fixed</u> lines, say $x=x_0$ or $y=y_0$, then the corresponding eigenfunctions would be discontinuous at fixed points (see example that I gave in class: $K(x,y)=1$, $0\le x,y\le\frac{1}{2}$ and $K(x,y)=2$ otherwise, where $0\le x,y\le 1$ in the entire region of definition.)

(19) $$u(x)=f(x)+\lambda\int_0^1 dy\; K(x,y)\,u(y), \quad \text{where} \quad K(x,y)=\begin{cases}3, & 0\le y<x\le 1,\\ 2, & 0\le x<y\le 1.\end{cases}$$

(a) Homogeneous equation:

$$u(x)=\lambda\int_0^x dy\; 3\,u(y)+\lambda\int_x^1 dy\; 2\,u(y) = 2\lambda\int_0^1 dy\; u(y)+\lambda\int_0^x dy\; u(y)$$

$$\Rightarrow u'(x)=\lambda u(x) \quad\Rightarrow\quad u(x)=C_1\, e^{\lambda x}, \quad 0\le x\le 1, \quad \overset{\text{const.}}{C_1}\neq 0.$$

From the original equation we get $\begin{cases} u(0)=2\lambda\int_0^1 dy\; u(y)\\ u(1)=3\lambda\int_0^1 dy\; u(y)\end{cases} \Rightarrow u(1)=\frac{3}{2}u(0).$

Hence, $C_1 e^{\lambda}=\frac{3}{2}C_1 \;\Rightarrow\; e^{\lambda}=\frac{3}{2}= e^{\ln(3/2)+i2n\pi}$, n: integer

$$\Rightarrow \lambda=\lambda_n=\ln\left(\tfrac{3}{2}\right)+i2n\pi, \quad n=0,\pm1,\pm2,\dots$$

The corresponding eigenfunctions are $u_n(x)=C_n\, e^{\lambda_n x}$. Take C_n: real.

Finally: $\quad u_n(x) = C_n \, e^{[\ln(\frac{3}{2}) + i2n\pi]x}$, $\quad n$: integer.

(b) Clearly, $K(x,y) \neq K(y,x)$: K is <u>not</u> symmetric.

$$K^T(x,y) = K(y,x) = \begin{cases} 3, & 0 \le x < y \le 1 \\ 2, & 0 \le y < x \le 1. \end{cases}$$

Homogeneous equation for K^T : $\quad v(x) = \lambda \int_0^1 dy \; K^T(x,y) \, v(y)$

$$\Rightarrow \; v(x) = 2\lambda \int_0^x dy \; v(y) + 3\lambda \int_x^1 dy \; v(y) = 2\lambda \int_0^1 dy \; v(y) + \lambda \int_x^1 dy \; v(y)$$

$\Rightarrow \; v'(x) = -\lambda v(x) \quad \Rightarrow \quad v(x) = F e^{-\lambda x}$, $\quad 0 \le x \le 1$, $\quad F$: const. $\neq 0$. Take F: real.

From original equation, $\quad \left. \begin{aligned} v(0) &= 3\lambda \int_0^1 dy \; v(y) \\ v(1) &= 2\lambda \int_0^1 dy \; v(y) \end{aligned} \right\} \Rightarrow \; v(1) = \frac{2}{3} v(0)$

$\Rightarrow \; F e^{-\lambda} = \frac{2}{3} F \quad \Rightarrow \quad \lambda = \lambda_n = \ln\left(\frac{3}{2}\right) + i2n\pi$, $\quad n$: integer $\quad$ (same as for K).

Eigenfunctions : $\quad v_m(x) = F_m \; e^{-[\ln(3/2) + i2m\pi]x}$, $\quad m$: integer.

(c) $$\int_0^1 dx \;\; u_n(x) \, v_m(x) = C_n F_m \int_0^1 dx \;\; e^{i2(n-m)\pi x} = 0 , \quad n \neq m.$$

(d) $$K(x,y) = \sum_{n=-\infty}^{\infty} \frac{u_n(x) \, v_n(y)}{\lambda_n} = \sum_{n=-\infty}^{\infty} \frac{e^{i2n(x-y)\pi + \ln(3/2)\cdot(x-y)}}{\ln(\frac{3}{2}) + i2n\pi} .$$

by taking $C_n = 1 = F_n$ so that $\int_0^1 dx \;\; u_n(x) \, v_m(x) = \delta_{nm}$.

In conlusion, $$K(x,y) = \sum_{n=-\infty}^{\infty} \frac{\exp\{[\ln(3/2) + i2n\pi](x-y)\}}{\ln(3/2) + i2n\pi} .$$

(20) In order to make more clear the connection to integral equations, allow me to take $\boxed{E\equiv\lambda}$.
Suppose that $\lambda=\lambda_n$ is any eigenvalue of $Au=\lambda u$, with $u=u_n$: eigenfunction.

$$Au(x)=\lambda u(x) \Rightarrow \int_0^1 dx\ u^*(x)\,A\,u(x)=\lambda\int_0^1 dx\cdot|u(x)|^2$$

$$\Rightarrow \int_0^1 dx\ u^*(x)\left\{-\frac{d^2u}{dx^2}+\int_0^1 dy\ xy\,u(y)\right\}=\lambda\int_0^1 dx\ |u(x)|^2$$

$$\Rightarrow \int_0^1 dx\ u^*(x)\left(-\frac{d^2u}{dx^2}\right)+\int_0^1 dx\int_0^1 dy\ u^*(x)\,u(y)\cdot xy=\lambda\int_0^1 dx\cdot|u(x)|^2$$

$$\Rightarrow \underbrace{-u^*(x)\frac{du}{dx}\Big|_0^1}_{\substack{=0,\ \text{because}\\ u^*(0)=0=u'(1)}}+\int_0^1 dx\cdot\left|\frac{du}{dx}\right|^2+\left|\int_0^1 dx\ u(x)\cdot x\right|^2=\lambda\int_0^1 dx\cdot|u(x)|^2$$

$$\Rightarrow \lambda=\left[\int_0^1 dx\ |u(x)|^2\right]^{-1}\left\{\int_0^1 dx\ \left|\frac{du}{dx}\right|^2+\left|\int_0^1 dx\ xu(x)\right|^2\right\}$$

It is inferred that $\lambda=\lambda_n>0$.

Suppose λ_n,λ_m are two different eigenvalues $(n\neq m)$, with eigenfunctions $u_n(x)$ and $u_m(x)$.

$$\left.\begin{array}{l} Au_n(x)=\lambda_n u_n(x) \Rightarrow u_m^*(x)\,A\,u_n(x)=\lambda_n\,u_m^*(x)\,u_n(x) \\[2ex] Au_m(x)=\lambda_m u_m(x) \Rightarrow Au_m^*(x)=\lambda_m u_m^*(x) \Rightarrow u_n(x)\,A\,u_m^*(x)=\lambda_m\,u_m^*(x)\,u_n(x) \end{array}\right\}$$

$$\Rightarrow u_m^*(x)\,A\,u_n(x)-u_n(x)\,A\,u_m^*(x)=(\lambda_n-\lambda_m)\ u_m^*(x)\,u_n(x).$$

$$\Rightarrow \int_0^1 dx\ \left[u_m^*(x)\,A\,u_n(x)-u_n(x)\,A\,u_m^*(x)\right]=(\lambda_n-\lambda_m)\int_0^1 dx\ u_m^*(x)\,u_n(x),$$

where
$$\int_0^1 dx\ u_m^*(x)\,A\,u_n(x)=\int_0^1 dx\ u_m^*(x)\left[-\frac{d^2u_n}{dx^2}+\int_0^1 dy\ xy\ u_n(y)\right]$$

$$=\int_0^1 dx\ \frac{du_m^*}{dx}\,\frac{du_n}{dx}+\int_0^1 dx\ x\,u_m^*(x)\cdot\int_0^1 dy\ y\,u_n(y).$$

Hence,

$$\int_0^1 dx\ \left[u_m^*(x)\,Au_n(x) - u_n(x)\,A\,u_m^*(x)\right]$$

$$= \int_0^1 dx\ \frac{du_m^*}{dx}\frac{du_n}{dx} + \int_0^1 dx\ x\,u_m^*(x)\cdot\int_0^1 dy\ y\,u_n(y)$$

$$- \int_0^1 dx\ \frac{du_n}{dx}\cdot\frac{du_m^*}{dx} - \int_0^1 dx\ x\,u_n(x).\int_0^1 dy\ y\,u_m^*(y) \quad = \quad 0.$$

$$\Longrightarrow \quad (\lambda_n-\lambda_m)\int_0^1 u_m^*(x)\ u_n(x)\ dx = 0 \quad \underset{\lambda_n\neq\lambda_m}{\Longrightarrow} \quad \int_0^1 u_m^*(x)\ u_n(x)\ dx = 0.$$

Since A is real $(A^* = A)$,

$$\left.\begin{aligned} Au &= \lambda u \\ \text{and}\quad Au^* &= \lambda u^* \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} A\,(u+u^*) &= \lambda\,(u+u^*) \\ A\,(u-u^*) &= \lambda\,(u-u^*) \end{aligned}\right. ,$$

i.e., if $u(x)$ is an eigenfunction, so is $u(x) \pm u^*(x)$, with the same eigenvalue. Because $u(x)+u^*(x)$ is real and $-i[u(x)-u^*(x)]$ is real, we can always take the eigenfunctions to be real without any loss of generality.

(b) This is a <u>tricky</u> question (my original solution was wrong, since I kept finding a non-symmetric kernel that depends on E !). I will use the Green's function method to find a λ-independent (E-), symmetric kernel.

Define $G(x,x')$ so that

$$\left\{\begin{aligned} -G_{xx}(x,x') &= \delta(x-x') \\ G(0,x') &= 0 = G_x(1,x') \end{aligned}\right. , \quad 0<x,x'<1.$$

It follows that

$$G(x,x') = \left\{\begin{aligned} &A_1x + B_1, \quad 0<x<x'<1 \\ &A_2x+B_2, \quad 0<x'<x<1 \end{aligned}\right. .$$

$$G(0,x')=0 \quad \Rightarrow \quad B_1=0$$
$$G_x(1,x')=0 \quad \Rightarrow \quad A_2=0.$$

Two further conditions on $G(x,x')$ are: $\begin{cases} G(x=x'^+,x') = G(x=x'^-,x') \\ G_x(x=x'^-,x') - G_x(x=x'^+,x') = 1 \end{cases}$

$$\Rightarrow \begin{cases} B_2 = A_1 x' \\ A_1 = 1 \end{cases} \Rightarrow \begin{cases} A_1 = 1 \\ B_2 = x' \end{cases} ; \qquad G(x,x') = \begin{cases} x, & 0<x<x'<1 \\ x', & 0<x'<x<1, \end{cases} = \min\{x,x'\}$$

The equation for $u(x)$ is cast in the following form

$$\begin{cases} -u''(x) = \overbrace{g(x) - b x}^{\text{"inhomogeneous term"}} \\ u(0) = 0 = u'(1) \end{cases}, \qquad \text{where } \underbrace{g(x) \equiv \lambda u(x)}_{\text{supposedly known for time being}}, \quad b \equiv \int_0^1 dy\, y u(y) = \text{const.}$$

By the Green's function method, it follows that

$$u(x) = \int_0^1 dx'\, G(x,x') \cdot [\, g(x') - b x' \,]$$
$$= \int_0^1 dx'\, G(x,x') g(x') - b \int_0^1 dx'\, x' G(x,x')$$

We use this equation to determine b in terms of integrals of G:

$$b = \int_0^1 dy\, y u(y) = \int_0^1 dy \cdot y \left[\int_0^1 dx'\, G(y,x') g(x') - b \int_0^1 dx'\, x' G(y,x') \right]$$
$$= \int_0^1 dy \int_0^1 dx'\, y g(x') G(y,x') - b \int_0^1 dy \int_0^1 dx'\, x'y\, G(y,x')$$
$$\Rightarrow \quad b = \frac{\int_0^1 dy \int_0^1 dx'\, y g(x') G(y,x')}{1 + \int_0^1 dy \int_0^1 dx'\, x'y\, G(y,x')} \equiv \frac{B}{1+A},$$

where $A \equiv \int_0^1 dy \int_0^1 dx' \underbrace{x'y\, G(y,x')}_{\substack{\text{symmetric}\\ \text{about } x'=y}} = 2\int_0^1 dy \int_0^y dx'\, x'y\, \min\{y,x'\} = 2\int_0^1 dy \int_0^y dx'\, x'^2 y$

$$= 2\int_0^1 dy \cdot y\, \frac{y^3}{3} = \frac{2}{3}\cdot\frac{1}{5} = \frac{2}{15} \Rightarrow A = \frac{2}{15},$$

$$B = \int_0^1 dy \int_0^1 dx'\, y g(x')\, G(y,x') = \int_0^1 dy \int_0^1 dx'\, g(x')\, yG(y,x') = \int_0^1 dx'\, g(x') \left[\int_0^1 dy\, y\, G(y,x') \right]$$

$$= \int_0^1 dx'\, g(x') \left[\int_0^{x'} dy\cdot y^2 + \int_{x'}^1 dy\cdot yx' \right] = \int_0^1 dx'\, g(x') \left[\frac{x'^3}{3} + x'\,\frac{1-x'^2}{2} \right] = \int_0^1 dx'\, g(x') \left(\frac{x'}{2} - \frac{x'^3}{6} \right)$$

Hence, $b = \frac{15}{17} \int_0^1 dx'\, g(x') \left(\frac{x'}{2} - \frac{x'^3}{6} \right) = \frac{15}{17} \int_0^1 dy\, g(y) \left(\frac{y}{2} - \frac{y^3}{6} \right)$

Accordingly, $u(x) = \int_0^1 dx'\, G(x,x')\, g(x') - b \underbrace{\int_0^1 dx' \cdot x' G(x,x')}$

$$= \int_0^1 dy\, G(x,y)\, g(y) - \left[\frac{15}{17} \int_0^1 dy\, g(y) \left(\frac{y}{2} - \frac{y^3}{6} \right) \right] \cdot \overbrace{\left(\frac{x}{2} - \frac{x^3}{6} \right)}$$

$$\Rightarrow u(x) = \int_0^1 dy \left[\min\{x,y\} - \frac{15}{17} \left(\frac{x}{2} - \frac{x^3}{6} \right) \left(\frac{y}{2} - \frac{y^3}{6} \right) \right] \underbrace{g(y)}_{\lambda u(y)}$$

$$\Rightarrow \boxed{u(x) = \lambda \int_0^1 dy \left[\min\{x,y\} - \frac{15}{17} \left(\frac{x}{2} - \frac{x^3}{6} \right) \left(\frac{y}{2} - \frac{y^3}{6} \right) \right] u(y)}$$

Notice that the kernel K of this equation is <u>symmetric</u>! This fact is consistent with $Au = \lambda u$ having <u>real</u> eigenvalues.

(c) The original equation, $Au = \lambda u$, reads as

$$u''(x) + \lambda u(x) = b x \quad , \qquad b \equiv \int_0^1 dy\; y u(y).$$

It follows that

$$u(x) = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x) + \frac{b}{\lambda}\, x \qquad ; \; (\lambda \neq 0).$$

- $u(0) = 0 \Rightarrow A = 0 \; ; \quad u(x) = B\sin(\sqrt{\lambda}\, x) + \frac{b}{\lambda}\, x$
- $u'(1) = 0 \Rightarrow B\sqrt{\lambda}\cos(\sqrt{\lambda}) + \frac{b}{\lambda} = 0 \Rightarrow b = -B\lambda^{3/2}\cos(\sqrt{\lambda}).$

$$b = \int_0^1 dy\; y u(y) = \int_0^1 dy\; y\left[B\sin(\sqrt{\lambda}\, y) + \frac{b}{\lambda} y\right] = -\frac{B}{\sqrt{\lambda}}\, y\cos(\sqrt{\lambda}\, y)\Big|_0^1 + \frac{B}{\sqrt{\lambda}}\int_0^1 dy\, \cos(\sqrt{\lambda}\, y)$$
$$+ \frac{b}{\lambda}\,\frac{1}{3} = -\frac{B}{\sqrt{\lambda}}\cos(\sqrt{\lambda}) + \frac{B}{\lambda}\sin(\sqrt{\lambda}) + \frac{b}{3\lambda}$$

$$\Rightarrow b\left(1 - \frac{1}{3\lambda}\right) = -\frac{B}{\sqrt{\lambda}}\cos(\sqrt{\lambda})\left[1 - \frac{1}{\sqrt{\lambda}}\tan(\sqrt{\lambda})\right]$$

$$\Rightarrow -B\lambda^{3/2}\cdot\cos(\sqrt{\lambda})\cdot\left(1 - \frac{1}{3\lambda}\right) = -\frac{B}{\sqrt{\lambda}}\cos(\sqrt{\lambda})\cdot\left[1 - \frac{1}{\sqrt{\lambda}}\tan(\sqrt{\lambda})\right]$$

Assume $\cos(\sqrt{\lambda}) \neq 0$ and $B \neq 0$:

$$\lambda^{3/2}\left(1 - \frac{1}{3\lambda}\right) = \frac{1}{\sqrt{\lambda}}\left[1 - \frac{1}{\sqrt{\lambda}}\tan(\sqrt{\lambda})\right]$$

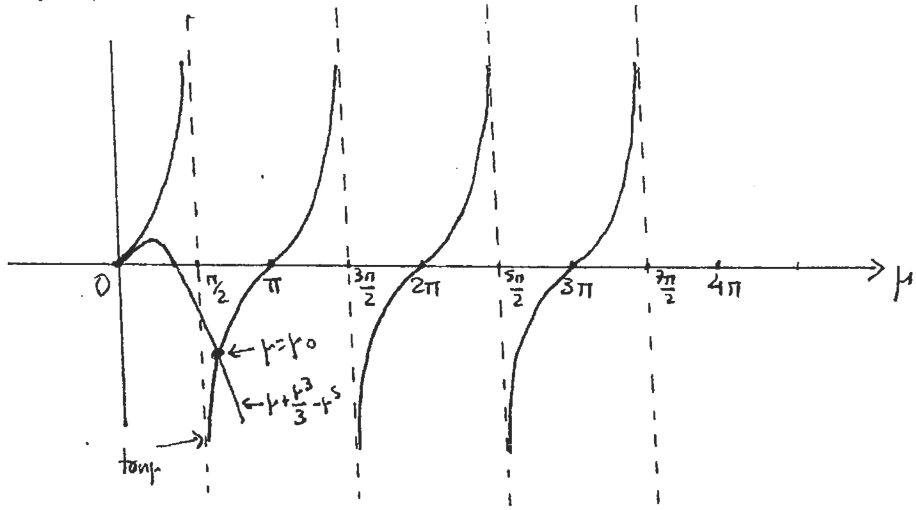
$$\Rightarrow \lambda^{5/2}\left(1 - \frac{1}{3\lambda}\right) = \sqrt{\lambda} - \tan(\sqrt{\lambda}). \qquad \text{Set } \sqrt{\lambda} \equiv \mu \rightarrow \mu = \lambda^2:$$

$$\mu^5\left(1 - \frac{1}{3\mu^2}\right) = \mu - \tan\mu \Leftrightarrow \tan\mu = \mu + \frac{\mu^3}{3} - \mu^5 \,, \qquad \mu > 0.$$

For $\underline{B = 0}$, one gets $b = 0$, and $u(x) \equiv 0$ (trivial).

For $\underline{\cos(\sqrt{\lambda}) = 0}$, one gets $b = 0$ and $\underline{B = 0}$. Hence, $B \neq 0 \neq \cos(\sqrt{\lambda})$

(d) My sketch is pretty awful and I hope that students will make a good estimate of the root!



The polynomial $P(\mu) = \mu + \frac{\mu^3}{3} - \mu^5$ has extrema at

$$P'(\mu)=0 \Rightarrow 1+\mu^2-5\mu^4=0 \underset{\mu:\text{real}}{\Rightarrow} \mu^2 = \frac{1+\sqrt{21}}{10} \Rightarrow \mu=\mu_{\pm} = \pm\sqrt{\frac{1+\sqrt{21}}{10}} \simeq \pm 0.7 .$$

At $\mu=\mu_+$, $P(\mu)$ has a maximum, and then it decreases rapidly to $-\infty$ as $\mu\to+\infty$.

Note that $P(0)=0$, $P'(0)=1$ and $\frac{d}{d\mu}(\tan\mu)\Big|_{\mu=0}=1$, while

$P''(0)=0$ and $\frac{d^2}{d\mu^2}(\tan\mu)\Big|_{\mu=0}=0$. More precisely,

$$\tan\mu \underset{\mu\to0+}{\sim} \mu + \frac{\mu^3}{3} + \frac{\mu^5}{8},$$

which means that $P(\mu)$ and $\tan\mu$ do not intersect in $(0, \pi/2)$.

I therefore expect $\mu=\mu_0$ to lie in $(\frac{\pi}{2}, \pi)$. Take $\boxed{\mu \simeq \frac{\pi}{2}}$

An estimate for $\lambda_0 = \mu_0^2$ is $\lambda \simeq \pi^2/4 \simeq 2.5$.

(e)

$$E_0 \equiv \lambda_0 = \min_{v\in D_A} \frac{\int_0^1 dx\, [v'(x)]^2 + [\int_0^1 dx\; x v(x)]^2}{\int_0^1 dx\; v(x)^2} .$$

For the trial function $v(x) = x(c-x)$ to belong to D_A, we have to choose $c=2$.

Then $v(x) = x(2-x)$, $v(0) = 0$, $v'(x) = 2-2x \Rightarrow v'(1) = 0$.

$$\int_0^1 dx \; v(x)^2 = \int_0^1 dx \; x^2(2-x)^2 = \int_0^1 dx \; x^2(4+x^2-4x) = 4\,\tfrac{1}{3} + \tfrac{1}{5} - 4\cdot\tfrac{1}{4}$$

$$= \frac{4}{3} + \frac{1}{5} - 1 = \frac{20+3-15}{15} = \frac{8}{15},$$

$$\int_0^1 dx \; [v'(x)]^2 = \int_0^1 dx \; 4(1-x)^2 = 4\int_0^1 dx \; (1+x^2-2x) = 4\left(1+\tfrac{1}{3} - 2\cdot\tfrac{1}{2}\right) = 4/3,$$

$$\int_0^1 dx \; x\,v(x) = \int_0^1 dx \; x^2(2-x) = 2\,\tfrac{1}{3} - \tfrac{1}{4} = \tfrac{2}{3} - \tfrac{1}{4} = \frac{8-3}{12} = 5/12$$

$$\lambda_0 \leq \frac{4/3 + 25/144}{8/15} = \frac{\frac{192+25}{144}}{\frac{8}{15}} = \frac{\overset{5}{\cancel{15}}\cdot 217}{8\cdot\underset{48}{\cancel{144}}} = \frac{1085}{384} \simeq 2.82$$

According to the trace inequality,

$$\lambda_0 \geq \left[\int_0^1 dx \; K_2(x,x)\right]^{-1/2},$$

where $$K_2(x,y) = \int_0^1 d\xi \; K(x,\xi)\,K(\xi,y) = \int_0^1 d\xi \; K(x,\xi)\,K(y,\xi) \quad \text{for } K\text{: symmetric},$$

$$\int_0^1 dx \; K_2(x,x) = \int_0^1 dx \int_0^1 d\xi \; K(x,\xi)^2 = \|K\|^2,$$

and $$K(x,y) = \min\{x,y\} - \frac{15}{17}\left(\frac{x}{2} - \frac{x^3}{6}\right)\left(\frac{y}{2} - \frac{y^3}{6}\right) \quad \text{from p. 11.}$$

$$\|K\|^2 = \int_0^1 dx \int_0^1 dy \; |K(x,y)|^2 = 2\int_0^1 dx \int_0^x dy \cdot \left[y - \frac{15}{17}\left(\frac{x}{2} - \frac{x^3}{6}\right)\left(\frac{y}{2} - \frac{y^3}{6}\right)\right]^2$$

$$= 2\int_0^1 dx \int_0^x dy \; \left[y^2 + \frac{15^2}{17^2}\left(\frac{x}{2} - \frac{x^3}{6}\right)^2\left(\frac{y}{2} - \frac{y^3}{6}\right)^2 - \frac{30}{17}\left(\frac{x}{2} - \frac{x^3}{6}\right)\left(\frac{y^2}{2} - \frac{y^4}{6}\right)\right]$$

$$= 2\int_0^1 dx\ \frac{x^3}{3} + 2\cdot\frac{15^2}{17^2}\int_0^1 dx\ \left(\frac{x}{2}-\frac{x^3}{6}\right)^2\int_0^x dy\ \left(\frac{y}{2}-\frac{y^3}{6}\right)^2 - \frac{60}{17}\int_0^1 dx\ \left(\frac{x}{2}-\frac{x^3}{6}\right)\int_0^x dy\ \left(\frac{y^2}{2}-\frac{y^4}{6}\right)$$

$$= \frac{2}{3}\cdot\frac{1}{4} + 2\cdot\frac{15^2}{17^2}\int_0^1 dx\cdot\left(\frac{x}{2}-\frac{x^3}{6}\right)^2\cdot\left(\frac{x^3}{12}+\frac{x^7}{36\cdot 7}-\frac{x^5}{30}\right) - \frac{60}{17}\int_0^1 dx\ \left(\frac{x}{2}-\frac{x^3}{6}\right)\left(\frac{x^3}{6}-\frac{x^5}{30}\right)$$

$$= \frac{1}{6} + 2\cdot\frac{15^2}{17^2}\int_0^1 dx\ \left(\frac{x^2}{4}+\frac{x^6}{36}-\frac{x^4}{6}\right)\cdot\left(\frac{x^3}{12}+\frac{x^7}{36\cdot 7}-\frac{x^5}{30}\right) - \frac{60}{17}\int_0^1 dx\ \left(\frac{x^4}{12}-\frac{x^6}{60}-\frac{x^6}{36}+\frac{x^8}{180}\right)$$

$$= \frac{1}{6} + 2\cdot\frac{15^2}{17^2}\int_0^1 dx\left(\frac{x^5}{48}+\frac{x^9}{4\cdot 36\cdot 7}-\frac{x^7}{120}+\frac{x^9}{36\cdot 12}+\frac{x^{13}}{36^2\cdot 7}-\frac{x^{11}}{36\cdot 30}-\frac{x^7}{6\cdot 12}-\frac{x^{11}}{36\cdot 6\cdot 7}+\frac{x^9}{6\cdot 30}\right)$$
$$- \frac{60}{17}\int_0^1 dx\ \left(\frac{x^4}{12}-\frac{x^6}{60}-\frac{x^6}{36}+\frac{x^8}{180}\right)$$

$$= \frac{1}{6} + 2\cdot\frac{15^2}{17^2}\left(\frac{1}{6\cdot 48}+\frac{1}{10\cdot 4\cdot 36\cdot 7}-\frac{1}{8\cdot 120}+\frac{1}{10\cdot 36\cdot 12}+\frac{1}{14\cdot 36^2\cdot 7}-\frac{1}{12\cdot 36\cdot 30}-\frac{1}{8\cdot 6\cdot 12}-\frac{1}{12\cdot 36\cdot 42}\right.$$
$$\left.+\frac{1}{60\cdot 30}\right) - \frac{60}{17}\left(\frac{1}{60}-\frac{1}{420}-\frac{1}{7\cdot 36}+\frac{1}{9\cdot 180}\right) \simeq 0.13034$$

Hence, $\lambda_0 \geq \dfrac{1}{\sqrt{0.13034}} \simeq 2.77$

Therefore, $2.77 \leq \lambda_0 < 2.82$