18.307: Integral Equations

M.I.T. Department of Mathematics

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Solutions to Set # 1

① (a) $$u(x) = 1 + \lambda \int_0^1 dy\ (x+y)\, u(y)$$

Set $\alpha = \int_0^1 dy\ u(y)$, $\beta = \int_0^1 dy\ y\, u(y)$

$$u(x) = 1 + \lambda(\alpha x + \beta)$$

$$\alpha = \int_0^1 dy\ [1 + \lambda(\alpha y + \beta)] = 1 + \lambda\left(\frac{\alpha}{2} + \beta\right) \implies \left(1 - \frac{\lambda}{2}\right)\alpha - \lambda\beta = 1$$

$$\beta = \int_0^1 dy\ [y + \lambda(\alpha y^2 + \beta y)] = \frac{1}{2} + \lambda\left(\frac{\alpha}{3} + \frac{\beta}{2}\right) \implies -\frac{\lambda}{3}\alpha + \left(1 - \frac{\lambda}{2}\right)\beta = \frac{1}{2}$$

The determinant of this system is

$$D = \begin{vmatrix} 1 - \lambda/2 & -\lambda \\ -\lambda/3 & 1 - \lambda/2 \end{vmatrix} = (1 - \lambda/2)^2 - \lambda^2/3 = \left(1 - \frac{\lambda}{2} - \frac{\lambda}{\sqrt{3}}\right)\left(1 - \frac{\lambda}{2} + \frac{\lambda}{\sqrt{3}}\right)$$

(i) For $\underline{D \neq 0} \iff \lambda \neq \dfrac{1}{\frac{1}{2} + \frac{1}{\sqrt{3}}} = \dfrac{2\sqrt{3}}{2+\sqrt{3}}$ and $\lambda \neq \dfrac{1}{\frac{1}{2} - 1/\sqrt{3}} = -\dfrac{2\sqrt{3}}{2 - \sqrt{3}}$

the system has the unique solution

$$\alpha = \frac{\begin{vmatrix} 1 & -\lambda \\ 1/2 & 1 - \lambda/2 \end{vmatrix}}{\left[1 - \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\lambda\right]\cdot\left[1 - \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\lambda\right]} = \frac{1 - \lambda/2 + \lambda/2}{\left[1 - \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\lambda\right]\cdot\left[1 - \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\lambda\right]} = \frac{1}{\left[\left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\lambda - 1\right]\cdot\left[\left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\lambda - 1\right]},$$

$$\beta = \frac{\begin{vmatrix} 1 - \lambda/2 & 1 \\ -\lambda/3 & 1/2 \end{vmatrix}}{\left[\left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\lambda - 1\right]\cdot\left[\left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\lambda - 1\right]} = \frac{\frac{1}{2}\left(\frac{\lambda}{6} + 1\right)}{\left[\left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\lambda - 1\right]\cdot\left[\left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\lambda - 1\right]}$$

(ii) For $D = 0 \iff \lambda = \dfrac{2\sqrt{3}}{2+\sqrt{3}}$ or $-\dfrac{2\sqrt{3}}{2-\sqrt{3}}$ the equation has $\underline{\text{no}}$ solutions.

since both of the determinants in the numerators for α and β are nonzero at these values of λ.

(b) For arbitrary integer n, the kernel of this equation is

$$K(x,y) = \frac{x^n - y^n}{x-y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + x^2y^{n-3} + xy^{n-2} + y^{n-1}$$

This is degenerate, i.e., it is a <u>finite</u> sum of products $g_i(x)\,h_i(y)$. Hence, the integral equation is

$$u(x) = 1 + \lambda \int_0^1 dy\ [x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + x^2y^{n-3} + xy^{n-2} + y^{n-1}]\, u(y).$$

We set $\alpha_1 = \int_0^1 dy\ u(y)$, $\alpha_2 = \int_0^1 dy\ y\,u(y)$, $\dots$, $\alpha_n = \int_0^1 dy\ y^{n-1}\, u(y)$.

Then $u(x)$ is

$$u(x) = 1 + \lambda\,(\alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \alpha_3 x^{n-3} + \dots + \alpha_{n-2}x^2 + \alpha_{n-1}x + \alpha_n) = 1 + \lambda \sum_{j=1}^{n} a_j x^{n-j}$$

α_p $(p = 1, 2, \dots, n)$ satisfy a system of linear equations:

$$\alpha_p = \int_0^1 dy\ y^{p-1}\left[1 + \lambda \sum_{j=1}^{n} a_j y^{n-j}\right] = \frac{1}{p} + \lambda \sum_{j=1}^{n} a_j \frac{1}{n-j+p} \qquad (p = 1, 2, \dots, n)$$

$$\Longrightarrow \quad a_p = \frac{1}{p} + \lambda \sum_{j=1}^{n} \frac{a_j}{n-j+p} \quad \Longrightarrow \quad \sum_{j=1}^{n}\left(\delta_{jp} - \frac{\lambda}{n-j+p}\right) a_j = \frac{1}{p}$$

$p = 1, 2, \dots, n.$

- This system has a <u>unique</u> solution if

  <u>$\det(I - \lambda \mathbb{K}) \neq 0$</u>, where $\mathbb{K}$: $\mathbb{K}_{pj} = \dfrac{1}{p-j+n}$: matrix elements of $\mathbb{K}$.
  I: identity matrix
  $p, j = 1, 2, \dots, n$.

- The system has <u>no solution</u> (<u>why</u>?) if

  <u>$\det(I - \lambda \mathbb{K}) = 0$</u>.

② $u(\theta) = 1 + \lambda \int_0^{2\pi} d\phi \; \sin(\phi-\theta)\, u(\phi) \quad , \quad 0 \le \theta < 2\pi.$

Notice that $u(\theta+2\pi) = 1+\lambda \int_0^{2\pi} d\phi \; \sin(\phi-\theta-2\pi)\, u(\phi) = 1+\lambda \int_0^{2\pi} d\phi \; \sin(\phi-\theta)\, u(\phi) = u(\theta)$,

i.e., the periodicity of $u(\theta)$ follows from the given equation.

Write $\quad \sin(\phi-\theta) = \sin\phi\cos\theta - \sin\theta\cos\phi$:

$$u(\theta) = 1 + \lambda \int_0^{2\pi} d\phi \; (\sin\phi\cos\theta - \sin\theta\cos\phi)\, u(\phi).$$

Set $\quad \alpha = \int_0^{2\pi} d\phi \; \sin\phi \; u(\phi), \quad \beta = \int_0^{2\pi} d\phi \; \cos\phi \; u(\phi)$

$$u(\theta) = 1 + \lambda\, (\alpha\cos\theta - \beta\sin\theta)$$

$$\left.\begin{aligned} \alpha &= \int_0^{2\pi} d\phi \; \sin\phi \; [1+\lambda(\alpha\cos\phi - \beta\sin\phi)] = -\lambda\beta \int_0^{2\pi} d\phi \; \sin^2\phi = -\lambda\beta\pi \;\Rightarrow\; \alpha + \lambda\beta\pi = 0 \\ \beta &= \int_0^{2\pi} d\phi \; \cos\phi \cdot [1+\lambda(\alpha\cos\phi - \beta\sin\phi)] = \lambda\alpha \int_0^{2\pi} d\phi \; \cos^2\phi = \lambda\alpha\pi \;\Rightarrow\; \lambda\alpha\pi - \beta = 0 \end{aligned}\right\}$$

$$\rightarrow \quad \beta = \lambda\pi\,\alpha = \lambda\pi\,(-\lambda\beta\pi) = -(\lambda\pi)^2\beta \;\Rightarrow\; \beta\,[1+(\lambda\pi)^2] = 0 \;\Rightarrow\; \begin{cases} \beta = 0 \\ \text{or} \\ \lambda\pi = \pm i \to \lambda = \frac{\pm i}{\pi}. \end{cases}$$

- For $\underline{\beta = 0}^{\,\lambda \ne \frac{\pm i}{\pi}} \Rightarrow \underline{\alpha = 0}$ and the only solution is $\underline{u(\theta) = 1}$, for $\lambda \ne \pm\frac{i}{\pi}$.
- For $\underline{\beta \ne 0} \Rightarrow \lambda = \pm\frac{i}{\pi}$, $\alpha = -\lambda\beta\pi = -\beta\pi\,\frac{\pm i}{\pi} = \mp i\beta$, $\beta = \pm i\alpha$

and the equation has infinitely many solutions: $u(\theta) = 1 \pm \frac{i}{\pi}\alpha\,(\cos\theta \mp i\sin\theta)$

$$= 1 \pm \frac{i}{\pi}\alpha\; e^{\mp i\theta}, \quad \alpha: \text{arbitrary}$$

The kernel of the equation has eigenvalues that are imaginary, $\lambda = \pm\frac{i}{\pi}$.

These are found by solving $\quad u(\theta) = \lambda \int_0^{2\pi} d\phi \; \sin(\phi-\theta)\, u(\phi).$

③ $$u(x) = 1+\lambda \int_0^1 dy \ u(y)^2$$

$$\alpha = \int_0^1 dy \ u(y)^2 , \qquad u(x) = 1+\lambda\alpha : \text{ constant.}$$

$$\alpha = \int_0^1 dy \ (1+\lambda\alpha)^2 \Rightarrow \alpha = (1+\lambda\alpha)^2 , \qquad \lambda^2\alpha^2 + (2\lambda-1)\alpha + 1 = 0.$$

$$\alpha = \frac{1-2\lambda \pm \sqrt{1-4\lambda}}{2\lambda^2}$$

$$u(x) = 1+ \frac{1-2\lambda \pm \sqrt{1-4\lambda}}{2\lambda} : \text{ 2 solutions}$$

For $\lambda \le 1/4$, both solutions are real.
For $\lambda > 1/4$, both solutions are complex, not real.

If we require that $u(x)$: real, then $\lambda = 1/4$ is a bifurcation point.

As $\lambda \to 0$, one of the solutions (with the upper sign) blows up:

$$u^+(x) = 1+ \frac{1-2\lambda+\sqrt{1-4\lambda}}{2\lambda} \underset{\lambda\to 0}{\sim} 1+ \frac{1-2\lambda+1-2\lambda}{2\lambda} = 1+\frac{1-2\lambda}{\lambda} = \frac{1}{\lambda} - 1 \sim \frac{1}{\lambda}$$

$$u^-(x) = 1+ \frac{1-2\lambda-\sqrt{1-4\lambda}}{2\lambda} \underset{\lambda\to 0}{\sim} 1+ \frac{1-2\lambda-(1-2\lambda)}{2\lambda} = 1$$

$\lambda = 0$ is a singular point of this equation and is not considered as a bifurcation point.