Method of Green's Functions

18.303 Linear Partial Differential Equations

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We introduce another powerful method of solving PDEs. First, we need to consider some preliminary definitions and ideas.

1 Preliminary ideas and motivation

1.1 The delta function

Ref: Guenther & Lee §10.5, Myint-U & Debnath §10.1

Definition [Delta Function] The δ-function is defined by the following three properties,

$$\delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0, \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$$

where f is continuous at $x = a$. The last is called the *sifting property* of the δ-function.

To make proofs with the δ-function more rigorous, we consider a δ-sequence, that is, a sequence of functions that converge to the δ-function, at least in a pointwise sense. Consider the sequence

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-(nx)^2}$$

Note that

$$\int_{-\infty}^{\infty} \delta_n(x)\, dx = \frac{2n}{\sqrt{\pi}} \int_0^{\infty} e^{-(nx)^2} dx = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz = \operatorname{erf}(\infty) = 1$$

Definition [2D Delta Function] The 2D δ-function is defined by the following three properties,

$$\delta(x,y) = \begin{cases} 0, & (x,y) \neq 0, \\ \infty, & (x,y) = 0, \end{cases}$$

$$\int\int \delta(x,y)\, dA = 1,$$

$$\int\int f(x,y)\,\delta(x-a, y-b)\, dA = f(a,b).$$

1.2 Green's identities

Ref: Guenther & Lee §8.3

Recall that we derived the identity

$$\int\int_D (G\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla G)\, dA = \int_C (G\mathbf{F})\cdot\hat{\mathbf{n}} dS \tag{1}$$

for any scalar function G and vector valued function $\mathbf{F}$. Setting $\mathbf{F} = \nabla u$ gives what is called Green's First Identity,

$$\int\int_D \left(G\nabla^2 u + \nabla u\cdot\nabla G\right) dA = \int_C G(\nabla u\cdot\hat{\mathbf{n}})\, dS \tag{2}$$

Interchanging G and u and subtracting gives Green's Second Identity,

$$\int\int_D \left(u\nabla^2 G - G\nabla^2 u\right) dA = \int_C (u\nabla G - G\nabla u)\cdot\hat{\mathbf{n}} dS. \tag{3}$$

2 Solution of Laplace and Poisson equation

Ref: Guenther & Lee, §5.3, §8.3, Myint-U & Debnath §10.2 – 10.4

Consider the BVP

$$\begin{aligned} \nabla^2 u &= F \quad \text{in} \quad D, \\ u &= f \quad \text{on} \quad C. \end{aligned} \tag{4}$$

Let (x,y) be a fixed arbitrary point in a 2D domain D and let (ξ,η) be a variable point used for integration. Let r be the distance from (x,y) to (ξ,η),

$$r = \sqrt{(\xi-x)^2 + (\eta-y)^2}.$$

Considering the Green's identities above motivates us to write

$$\begin{aligned} \nabla^2 G &= \delta(\xi-x, \eta-y) = \delta(r) \quad \text{in} \quad D, \\ G &= 0 \quad \text{on} \quad C. \end{aligned} \tag{5}$$

The notation $\delta(r)$ is short for $\delta(\xi - x, \eta - y)$. Substituting (4) and (5) into Green's second identity (3) gives

$$u(x,y) - \int\int_D GF dA = \int_C f \nabla G \cdot \hat{\mathbf{n}} dS$$

Rearranging gives

$$u(x,y) = \int\int_D GF dA + \int_C f \nabla G \cdot \hat{\mathbf{n}} dS \tag{6}$$

Therefore, if we can find a G that satisfies (5), we can use (6) to find the solution $u(x,y)$ of the BVP (4). The advantage is that finding the Green's function G depends only on the area D and curve C, not on F and f.

Note: this method can be generalized to 3D domains.

2.1 Finding the Green's function

To find the Green's function for a 2D domain D, we first find the simplest function that satisfies $\nabla^2 v = \delta(r)$. Suppose that $v(x,y)$ is axis-symmetric, that is, $v = v(r)$. Then

$$\nabla^2 v = v_{rr} + \frac{1}{r} v_r = \delta(r)$$

For $r > 0$,

$$v_{rr} + \frac{1}{r} v_r = 0$$

Integrating gives

$$v = A \ln r + B$$

For simplicity, we set $B = 0$. To find A, we integrate over a disc of radius ε centered at (x,y), D_ε,

$$1 = \int\int_{D_\varepsilon} \delta(r) dA = \int\int_{D_\varepsilon} \nabla^2 v dA$$

From the Divergence Theorem, we have

$$\int\int_{D_\varepsilon} \nabla^2 v dA = \int_{C_\varepsilon} \nabla v \cdot \mathbf{n} dS$$

where C_ε is the boundary of D_ε, i.e. a circle of circumference $2\pi\varepsilon$. Combining the previous two equations gives

$$1 = \int_{C_\varepsilon} \nabla v \cdot \mathbf{n} dS = \int_{C_\varepsilon} \left.\frac{\partial v}{\partial r}\right|_{r=\varepsilon} dS = \int_{C_\varepsilon} \frac{A}{\varepsilon} dS = 2\pi A$$

Hence

$$v(r) = \frac{1}{2\pi} \ln r$$

This is called the *fundamental solution* for the Green's function of the Laplacian on 2D domains. For 3D domains, the fundamental solution for the Green's function of the Laplacian is $-1/(4\pi r)$, where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$.

The Green's function for the Laplacian on 2D domains is defined in terms of the corresponding fundamental solution,

$$\begin{aligned} G(x, y; \xi, \eta) &= \frac{1}{2\pi} \ln r + h, \\ & \quad h \text{ is regular,} \\ \nabla^2 h &= 0, \quad (\xi, \eta) \in D, \\ G &= 0 \quad (\xi, \eta) \in C. \end{aligned}$$

The term "regular" means that h is twice continuously differentiable in (ξ, η) on D. Finding the Green's function G is reduced to finding a C^2 function h on D that satisfies

$$\begin{aligned} \nabla^2 h &= 0 \quad (\xi, \eta) \in D, \\ h &= -\frac{1}{2\pi} \ln r \quad (\xi, \eta) \in C. \end{aligned}$$

The definition of G in terms of h gives the BVP (5) for G. Thus, for 2D regions D, finding the Green's function for the Laplacian reduces to finding h.

2.2 Examples

Ref: Myint-U & Debnath §10.6

(i) Full plane $D = \mathbb{R}^2$. There are no boundaries so $h = 0$ will do, and

$$G = \frac{1}{2\pi} \ln r = \frac{1}{4\pi} \ln \left[(\xi - x)^2 + (\eta - y)^2 \right]$$

(ii) Half plane $D = \{(x, y) : y > 0\}$. We find G by introducing what is called an "image point" $(x, -y)$ corresponding to (x, y). Let r be the distance from (ξ, η) to (x, y) and r' the distance from (ξ, η) to the image point $(x, -y)$,

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2}, \qquad r' = \sqrt{(\xi - x)^2 + (\eta + y)^2}$$

We add

$$h = -\frac{1}{2\pi} \ln r' = -\frac{1}{2\pi} \ln \sqrt{(\xi - x)^2 + (\eta + y)^2}$$

to G to make $G = 0$ on the boundary. Since the image point $(x, -y)$ is NOT in D, then h is regular for all points $(\xi, \eta) \in D$, and satisfies Laplace's equation,

$$\nabla^2 h = \frac{\partial^2 h}{\partial \xi^2} + \frac{\partial^2 h}{\partial \eta^2} = 0$$

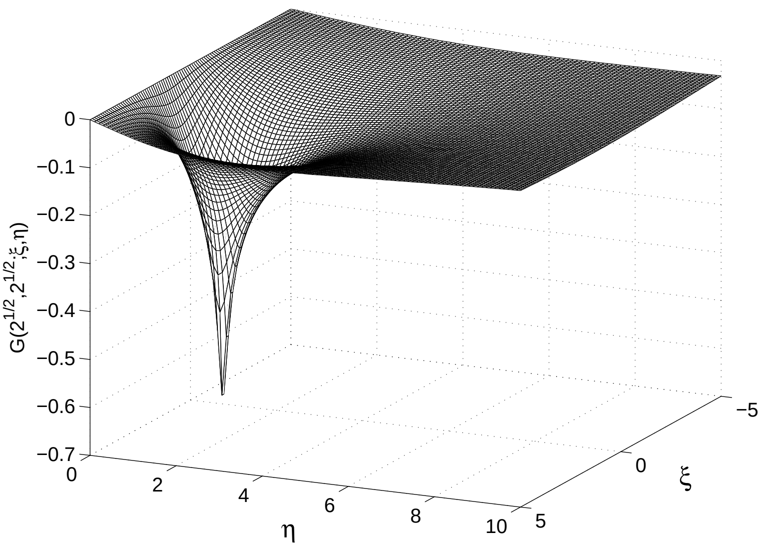

Figure 1: Plot of the Green's function $G(x, y; \xi, \eta)$ for the Laplacian operator in the upper half plane, for $(x, y) = (\sqrt{2}, \sqrt{2})$.

for $(\xi, \eta) \in D$. Writing things out fully, we have

$$G = \frac{1}{2\pi} \ln r + h = \frac{1}{2\pi} \ln r - \frac{1}{2\pi} \ln r' = \frac{1}{2\pi} \ln \frac{r}{r'} = \frac{1}{4\pi} \ln \frac{(\xi - x)^2 + (\eta - y)^2}{(\xi - x)^2 + (\eta + y)^2} \tag{7}$$

$G(x, y; \xi, \eta)$ is plotted in the upper half plane in Figure 1 for $(x, y) = \left(\sqrt{2}, \sqrt{2}\right)$. Note that $G \to -\infty$ as $(\xi, \eta) \to (x, y)$. Also, notice that $G < 0$ everywhere and $G = 0$ on the boundary $\eta = 0$. These are, in fact, general properties of the Green's function. The Green's function $G(x, y; \xi, \eta)$ acts like a weighting function for (x, y) and neighboring points in the plane. The solution u at (x, y) involves integrals of the weighting $G(x, y; \xi, \eta)$ times the boundary condition $f(\xi, \eta)$ and forcing function $F(\xi, \eta)$.

On the boundary C, $\eta = 0$, so that $G = 0$ and

$$\nabla G \cdot \mathbf{n} = -\left.\frac{\partial G}{\partial \eta}\right|_{\eta=0} = \frac{1}{\pi} \frac{y}{(\xi - x)^2 + y^2}$$

The solution of the BVP (6) with $F = 0$ on the upper half plane D can now be written as, from (6),

$$u(x, y) = \int_C f \nabla G \cdot \hat{\mathbf{n}} dS = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(\xi - x)^2 + y^2} d\xi,$$

which is the same as we found from the Fourier Transform, on page 13 of fourtran.pdf.

(iii) Upper right quarter plane $D = \{(x, y) : x > 0, y > 0\}$. We use the image points $(x, -y)$, $(-x, y)$ and $(-x, -y)$,

$$\begin{aligned} G &= \frac{1}{2\pi}\ln\sqrt{(\xi - x)^2 + (\eta - y)^2} - \frac{1}{2\pi}\ln\sqrt{(\xi - x)^2 + (\eta + y)^2} \\ &\quad -\frac{1}{2\pi}\ln\sqrt{(\xi + x)^2 + (\eta - y)^2} + \frac{1}{2\pi}\ln\sqrt{(\xi + x)^2 + (\eta + y)^2} \end{aligned} \tag{8}$$

For $(\xi, \eta) \in C = \partial D$ (the boundary), either $\xi = 0$ or $\eta = 0$, and in either case, $G = 0$. Thus $G = 0$ on the boundary of D. Also, the second, third and fourth terms on the r.h.s. are regular for $(\xi, \eta) \in D$, and hence the Laplacian $\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$ of each of these terms is zero. The Laplacian of the first term is $\delta(r)$. Hence $\nabla^2 G = \delta(r)$. Thus (8) is the Green's function in the upper half plane D.

For $(\xi, \eta) \in C = \partial D$ (the boundary),

$$\begin{aligned} \int_C f\nabla G \cdot \hat{\mathbf{n}} dS &= \int_\infty^0 f(0, \eta)\left(-\left.\frac{\partial G}{\partial \xi}\right|_{\xi=0}\right) d\eta + \int_0^\infty f(\xi, 0)\left(-\left.\frac{\partial G}{\partial \eta}\right|_{\eta=0}\right) d\xi \\ &= \int_0^\infty f(0, \eta)\left.\frac{\partial G}{\partial \xi}\right|_{\xi=0} d\eta - \int_0^\infty f(\xi, 0)\left.\frac{\partial G}{\partial \eta}\right|_{\eta=0} d\xi \end{aligned}$$

Note that

$$\begin{aligned} \left.\frac{\partial G}{\partial \xi}\right|_{\xi=0} &= -\frac{4yx\eta}{\pi\left(x^2 + (y + \eta)^2\right)\left(x^2 + (y - \eta)^2\right)} \\ \left.\frac{\partial G}{\partial \eta}\right|_{\eta=0} &= -\frac{4yx\xi}{\pi\left((x - \xi)^2 + y^2\right)\left((x + \xi)^2 + y^2\right)} \end{aligned}$$

The solution of the BVP (6) with $F = 0$ on the upper right quarter plane D and boundary condition $u = f$ can now be written as, from (6),

$$\begin{aligned} u(x, y) &= \int_C f\nabla G \cdot \hat{\mathbf{n}} dS \\ &= -\frac{4yx}{\pi}\int_0^\infty \frac{\eta f(0, \eta)}{\left(x^2 + (y + \eta)^2\right)\left(x^2 + (y - \eta)^2\right)} d\eta \\ &\quad + \frac{4yx}{\pi}\int_0^\infty \frac{\xi f(\xi, 0)}{\left((x - \xi)^2 + y^2\right)\left((x + \xi)^2 + y^2\right)} d\xi \end{aligned}$$

(iv) Unit disc $D = \{(x, y) : x^2 + y^2 \leq 1\}$. By some simple geometry, for each point $(x, y) \in D$, choosing the image point (x', y') along the same ray as (x, y) and a distance $1/\sqrt{x^2 + y^2}$ away from the origin guarantees that r/r' is constant along the circumference of the circle, where

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2}, \qquad r' = \sqrt{(\xi - x')^2 + (\eta - y')^2}.$$

[DRAW] Using the law of cosines, we obtain

$$\begin{aligned} r^2 &= \tilde{\rho}^2 + \rho^2 - 2\rho\tilde{\rho}\cos\left(\tilde{\theta} - \theta\right) \\ r'^2 &= \tilde{\rho}^2 + \frac{1}{\rho^2} - 2\frac{\tilde{\rho}}{\rho}\cos\left(\tilde{\theta} - \theta\right) \end{aligned}$$

where $\rho = \sqrt{x^2 + y^2}$, $\tilde{\rho} = \sqrt{\xi^2 + \eta^2}$ and θ, $\tilde{\theta}$ are the angles the rays (x, y) and (ξ, η) make with the horizontal. Note that for (ξ, η) on the circumference ($\xi^2 + \eta^2 = \tilde{\rho}^2 = 1$), we have

$$\frac{r^2}{r'^2} = \frac{1 + \rho^2 - 2\rho\cos\left(\tilde{\theta} - \theta\right)}{1 + \frac{1}{\rho^2} - 2\frac{1}{\rho}\cos\left(\tilde{\theta} - \theta\right)} = \rho^2, \qquad \tilde{\rho} = 1.$$

Thus the Green's function for the Laplacian on the 2D disc is

$$G\left(\xi, \eta; x, y\right) = \frac{1}{2\pi}\ln\frac{r}{r'\rho} = \frac{1}{4\pi}\ln\frac{\tilde{\rho}^2 + \rho^2 - 2\rho\tilde{\rho}\cos\left(\tilde{\theta} - \theta\right)}{\rho^2\tilde{\rho}^2 + 1 - 2\rho\tilde{\rho}\cos\left(\tilde{\theta} - \theta\right)}$$

Note that

$$\nabla G \cdot \hat{\mathbf{n}} = \left.\frac{\partial G}{\partial \tilde{\rho}}\right|_{\tilde{\rho}=1} = \frac{1}{2\pi}\frac{1 - \rho^2}{1 + \rho^2 - 2\rho\cos\left(\tilde{\theta} - \theta\right)}$$

Thus, the solution to the BVP (5) on the unit circle is (in polar coordinates),

$$\begin{aligned} u\left(\rho, \theta\right) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1 - \rho^2}{1 + \rho^2 - 2\rho\cos\left(\tilde{\theta} - \theta\right)}f\left(\tilde{\theta}\right)d\tilde{\theta} \\ &\quad + \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \ln\left(\frac{\tilde{\rho}^2 + \rho^2 - 2\rho\tilde{\rho}\cos\left(\tilde{\theta} - \theta\right)}{\rho^2\tilde{\rho}^2 + 1 - 2\rho\tilde{\rho}\cos\left(\tilde{\theta} - \theta\right)}\right)F\left(\tilde{\rho}, \tilde{\theta}\right)\tilde{\rho}d\tilde{\rho}d\theta \end{aligned}$$

The solution to Laplace's equation is found be setting $F = 0$,

$$u\left(\rho, \theta\right) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1 - \rho^2}{1 + \rho^2 - 2\rho\cos\left(\tilde{\theta} - \theta\right)}f\left(\tilde{\theta}\right)d\tilde{\theta}$$

This is called the Poisson integral formula for the unit disk.

2.3 Conformal mapping and the Green's function

Conformal mapping allows us to extend the number of 2D regions for which Green's functions of the Laplacian $\nabla^2 u$ can be found. We use complex notation, and let $\alpha = x + iy$ be a fixed point in D and let $z = \xi + i\eta$ be a variable point in D (what we're integrating over). If D is simply connected (a definition from complex analysis), then by the Riemann Mapping Theorem, there is a conformal map $w(z)$ (analytic and one-to-one) from D into the unit disk, which maps α to the origin, $w(\alpha) = 0$ and the boundary of D to the unit circle, $|w(z)| = 1$ for $z \in \partial D$ and $0 \leq |w(z)| < 1$ for $z \in D/\partial D$. The Greens function G is then given by

$$G = \frac{1}{2\pi} \ln |w(z)|$$

To see this, we need a few results from complex analysis. First, note that for $z \in \partial D$, $|w(z)| = 0$ so that $G = 0$. Also, since $w(z)$ is 1-1, $|w(z)| > 0$ for $z \neq \alpha$. Thus, we can write $w(z) = (z - \alpha)^n H(z)$ where $H(z)$ is analytic and nonzero in D. Since $w(z)$ is 1-1, $|w'(z)| > 0$ on D. Thus $n = 1$. Hence

$$w(z) = (z - \alpha) H(z)$$

and

$$G = \frac{1}{2\pi} \ln r + h$$

where

$$\begin{aligned} r &= |z - \alpha| = \sqrt{(\xi - x)^2 + (\eta - y)^2} \\ h &= \frac{1}{2\pi} \ln |H(z)| \end{aligned}$$

Since $H(z)$ is analytic and nonzero in D, then $(1/2\pi) \ln H(z)$ is analytic in D and hence its real part is harmonic, i.e. $h = \Re((1/2\pi) \ln H(z))$ satisfies $\nabla^2 h = 0$ in D. Thus by our definition above, G is the Green's function of the Laplacian on D.

Example 1. The half plane $D = \{(x, y) : y > 0\}$. The analytic function

$$w(z) = \frac{z - \alpha}{z - \alpha^*}$$

maps the upper half plane D onto the unit disc, where asterisks denote the complex conjugate. Note that $w(\alpha) = 0$ and along the boundary of D, $z = x$, which is equidistant from α and α^*, so that $|w(z)| = 1$. Points in the upper half plane $(y > 0)$ are closer to $\alpha = x + iy$, also in the upper half plane, than to $\alpha^* = x - iy$, in the lower half plane. Thus for $z \in D/\partial D$, $|w(z)| = |z - \alpha| / |z - \alpha^*| < 1$. The Green's function is

$$G = \frac{1}{2\pi} \ln |w(z)| = \frac{1}{2\pi} \ln \frac{|z - \alpha|}{|z - \alpha^*|} = \frac{1}{2\pi} \ln \frac{r}{r'}$$

which is the same as we derived before, Eq. (7).

3 Solution to other equations by Green's function

Ref: Myint-U & Debnath §10.5

The method of Green's functions can be used to solve other equations, in 2D and 3D. For instance, for a 2D region D, the problem

$$\begin{aligned} \nabla^2 u + u &= F \quad \text{in} \quad D, \\ u &= f \quad \text{on} \quad \partial D, \end{aligned}$$

has the fundamental solution

$$\frac{1}{4} Y_0(r)$$

where $Y_0(r)$ is the Bessel function of order zero of the second kind. The problem

$$\begin{aligned} \nabla^2 u - u &= F \quad \text{in} \quad D, \\ u &= f \quad \text{on} \quad \partial D, \end{aligned}$$

has fundamental solution

$$-\frac{1}{2\pi} K_0(r)$$

where $K_0(r)$ is the modified Bessel function of order zero of the second kind.

The Green's function method can also be used to solve time-dependent problems, such as the Wave Equation and the Heat Equation.