Lecture 9

Finished consideration of separability of $\nabla^2 u=\lambda u$ in a 2d box, from notes: discussed orthogonality of these eigenfunctions. Also showed that separability breaks down, in general, for non-constant coefficients in the box.

More separation of variables: cylindrical case of a cylinder of radius R with Dirichlet boundary conditions. Show that the Laplace eigenequation here is indeed separable into a function of θ multiplied by a function of r, satisfying separate 1d ODEs. Show that the θ dependence is sin(mθ) or cos(mθ) (or any linear combination), where m is an integer (in order to be periodic in θ). The r dependence satisfies a more complicated 2nd-order ODE that we can't solve by hand (even if you have taken 18.03).

At this point, it's more a historical question than a mathematical one: has someone solved this equation before, and if so is there a standard name (and normalization, etc) for the solutions? In fact, that is the case here (not surprisingly, since the Laplacian is so important): our r equation is an instance of **Bessel's equation**, and the solutions are called **Bessel functions**. The canonical two Bessel functions are J_m and Y_m: there is a standard convention defining the normalization, etcetera, of these, but the important thing for our purposes is that J is finite at r=0 and Y blows up at the origin. In Julia, SciPy, Matlab, and similar packages, these are supplied as built-in functions (e.g. besselj and bessely), and we use Julia to plot a few of them to get a feel for what they look like: basically, sinusoidal functions that are slowly decaying in r.

To get eigenfunctions, we have to impose boundary conditions. Finite-ness of the solution at r=0 means that we can only have $J_m(kr)$ solutions, and vanishing at r=R means that kR must be a root of J_m. We have to find these roots numerically, but this is easy to do, and we obtain a discrete set of eigenfunctions and eigenvalues.

From the general orthogonality of the Laplacian eigenfunctions, we can derive an orthogonality relation for Bessel functions, and by evaluating the integral numerically we can see that this orthogonality is indeed the case.

By looking at Bessel's equation asymptotically, we find that it reduces to sines and cosines for large r; more careful considerations show that it must actually reduce to sines and cosines multiplied by $1/\sqrt{r}$, and we can verify this from the plot. Conversely, for small r we show that it goes as either r^m (J_m) or $1/r^m$ (Y_m, except for m=1 where Y_0 is proportional to log r); this is why we have one finite solution and one divergent one at r=0. (There are many, many more properties of Bessel functions that one can derive analytically, but that is not our major concern here.)

Further reading: The Wikipedia page on Bessel functions has many plots, definitions, and properties, as well as links to standard reference works.