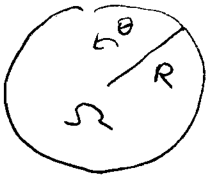
Lecture 8 : Cylindrical separability – Bessel functions

Ω = interior of circle in 2d



$u|_{\partial\Omega} = 0$

$\hat{A} = \nabla^2$: $\hat{A} = \hat{A}^*$, negative definite $\Rightarrow$ real $\lambda < 0$, $\perp$ eigenfunctions

separation ansatz : $\nabla^2 u = \lambda u \Rightarrow$ separable $\underline{u(r,\theta) = \rho(r)\,\tau(\theta)}$

$$\Rightarrow \nabla^2 u = \left[\frac{1}{r}\left(\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] u = \underbrace{\frac{1}{r}(r\rho')'\tau + \frac{1}{r^2}\rho\tau'' = \lambda\rho\tau}_{\times\ \frac{r^2}{\rho\tau}}$$

$$\Rightarrow \underbrace{\frac{r(r\rho')'}{\rho} - r^2\lambda}_{r \text{ only}} = \underbrace{-\frac{\tau''}{\tau}}_{\theta \text{ only}} = \# = +m^2$$

$\Rightarrow \tau'' = -\#\tau \Rightarrow \tau(\theta) =$ sines / cosines of $\sqrt{\#}\,\theta$ (or exp?)

$\Rightarrow \boxed{\tau(\theta) = \cos(m\theta) \text{ or } \sin(m\theta)}$ (or any linear comb.)

periodic : $\tau(\theta + 2\pi) = \tau(\theta) \Rightarrow \sqrt{\#} = m$ integer

$$\Rightarrow r(r\rho')' - (r^2\lambda + m^2)\rho = 0$$

$\lambda < 0 \Rightarrow$ let $\boxed{\lambda = -k^2}$ for some k

$$= \boxed{r^2\rho'' + r\rho' + (k^2r^2 - m^2)\rho = 0}$$

let $\boxed{\zeta = kr} \Rightarrow \boxed{\zeta^2\frac{d^2\rho}{d\zeta^2} + \zeta\frac{d\rho}{d\zeta} + (\zeta^2 - m^2)\rho = 0}$ "Bessel's equation" of order m

②

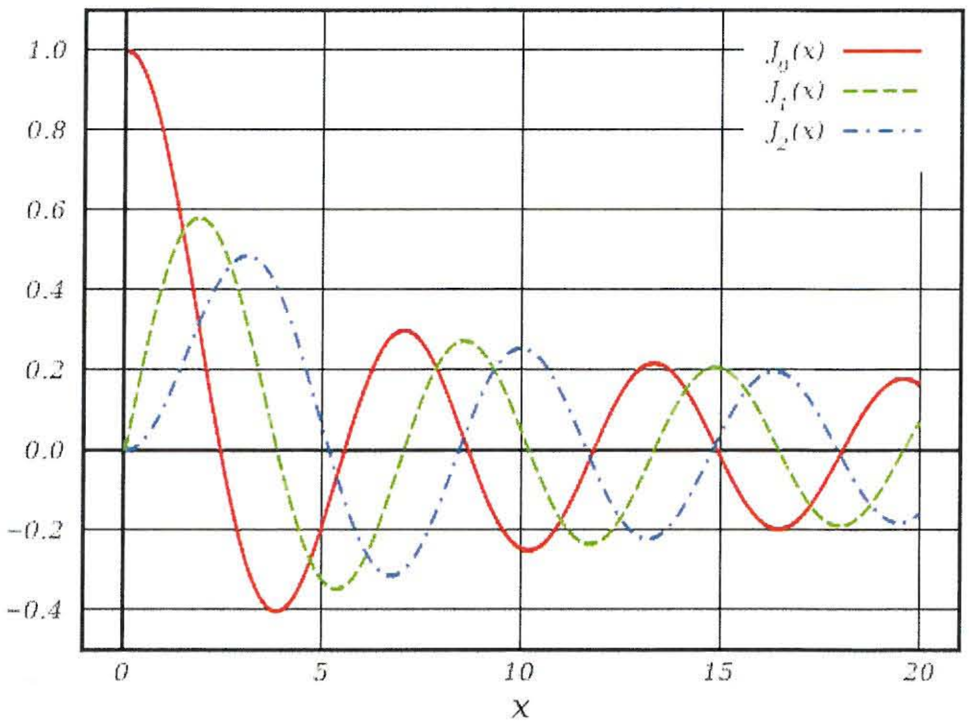
$\Rightarrow$ solutions must be some functions $J_m(\xi) = \boxed{J_m(kr) = \rho(r)}$

where J_m is "Bessel function of 1^{st} kind"

= "cylindrical analogue" of sine / cosine

– standard function, built into Matlab etc.

(Wikipedia)



oscillating, decaying function

... why?

* why oscillating? consider large r:

$$0 = r^2\rho'' + r\rho' + (k^2r^2 - m^2)\rho \approx r^2(\rho'' + k^2\rho)$$

$\Rightarrow \rho(r) \approx$ sin or cos of kr !

a little more carefully: suppose $\rho(r) \underset{kr \gg 1}{\approx} \cos(kr) \cdot r^p$ (or sin) for some unknown power p

$$\Longrightarrow 0 = r^2\rho'' + r\rho' + (k^2r^2 - m^2)\rho$$

$r^{p+2}\{$ $\quad \approx \cancel{-k^2r^2\cos(kr)\,r^p}$ $\qquad \cancel{+k^2r^2\cos(kr)\,r^p}$

$r^{p+1}\{$ $\quad -2kr^2\sin(kr)\,r^{p-1}\cdot p \qquad -kr\sin(kr)\,r^p$

$r^{p}\{$ $\quad +r^2\cos(kr)\,r^{p-2}\,p\cdot(p-1) \qquad +r\cos(kr)\,r^{p-1}\cdot p \qquad -m^2\cdot\cos\cdot r^p$

$kr \gg 1 \quad \propto -k\,r^{p+1}\sin(kr)\,(2p+1) \Longrightarrow \boxed{p = -\frac{1}{2}}$

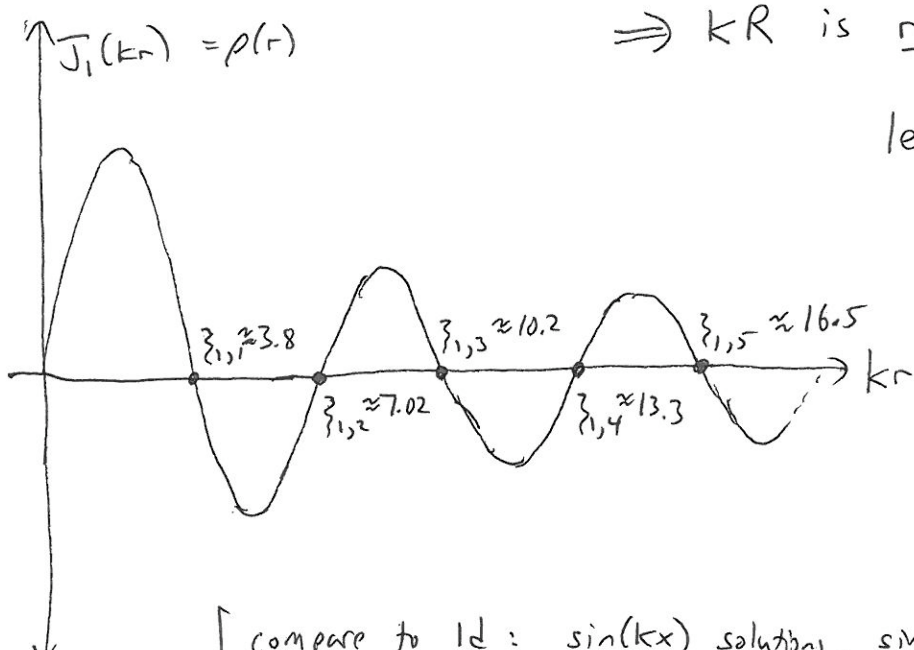
$$\Longrightarrow \underset{kr\gg1}{\rho(r)} \approx \frac{\cos \text{ or } \sin \text{ of } kr}{\sqrt{r}} \quad (\times \text{ some normalization})$$

[fancier analysis $\Longrightarrow \dots \Longrightarrow$ $\underset{kr \gg m^2}{J_m(kr)} \approx \sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{m\pi}{2} - \frac{\pi}{4}\right)$]

* Eigenvalues : $\quad \rho(R) = 0 = J_m(kR)$

$\Longrightarrow kR$ is root of J_m

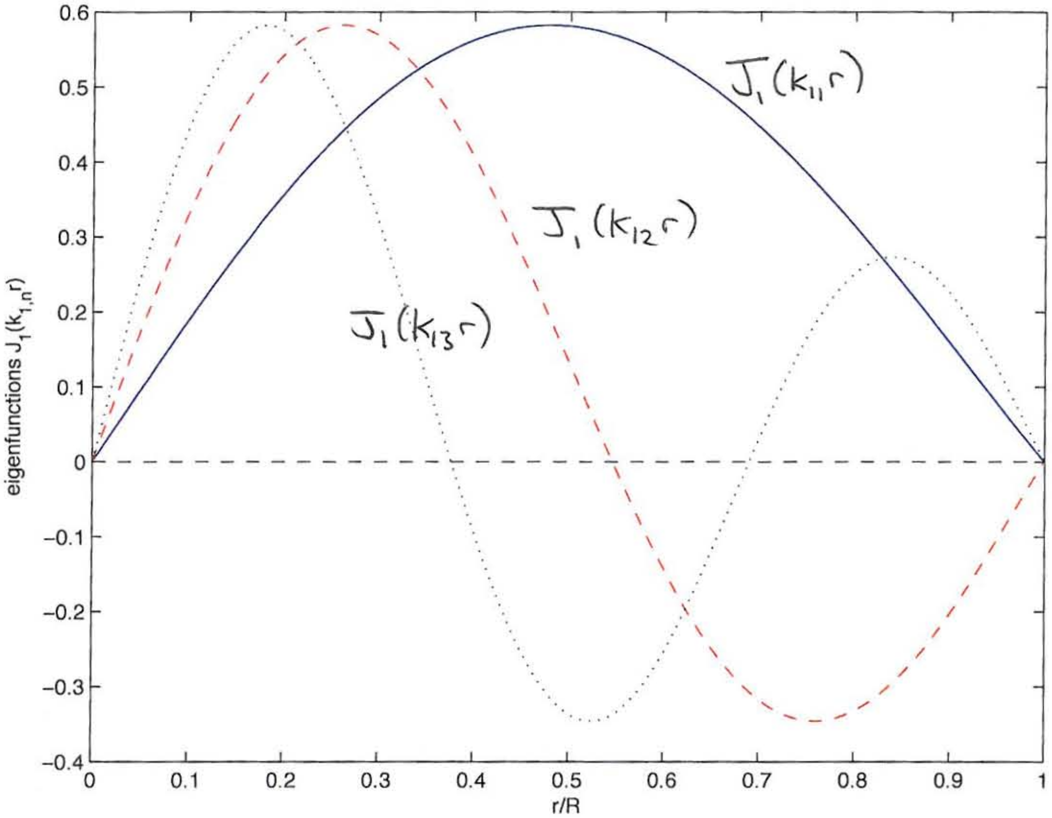
let n^{th} root of $J_m(\xi)$ $= \xi_{m,n}$



$$\Longrightarrow k_{m,n} = \frac{\xi_{m,n}}{R}$$

$$\Longrightarrow \boxed{\lambda = -\left(\frac{\xi_{m,n}}{R}\right)^2}$$

[compare to 1d : $\sin(kx)$ solutions, $\sin(kL) = 0$
$\Longrightarrow kL$ is root of $\sin() = n\pi$]



(radial) eigenfunctions $J_1(k_{1n} r)$

– compare to $\sin\left(\frac{n\pi r}{R}\right)$!

orthogonality: if $u_{m,n}$ and $u_{m',n'}$ are eigenfunctions with $\lambda_{mn} \neq \lambda_{m'n'}$ $\Rightarrow \langle u_{mn}, u_{m'n'} \rangle = 0$

$$\Rightarrow \int_0^{2\pi} d\theta \underbrace{\overset{\text{(or sines)}}{\cos(m\theta)\cos(m'\theta)}}_{0 \text{ if } m \neq m'} \underbrace{\int_0^R r\,dr\, J_m(k_{mn} r)\, J_m(k_{m'n'} r)}_{\text{must be 0 if } m=m'!\ n\neq n'}$$

$$\Rightarrow \int_0^R r\,dr\, J_m(k_{mn} r)\, J_m(k_{mn'} r)$$

(let $x = r/R$)

$$= R^2 \int_0^1 x\,dx\, J_m(\zeta_{mn} x)\, J_m(\zeta_{mn'} x) = 0$$

for $n \neq n'$ ($\Rightarrow$ must be oscillating!)

* Small-r behavior and the missing Bessel solution:

– Bessel's equation is 2nd order $\left(\frac{d^2}{dr^2}\right) \Rightarrow$ has 2 indep. sols!

consider behavior for $kr \ll 1$, suppose $\rho(r) \sim r^p$ for small r for some unknown power p

$$\Rightarrow 0 = r^2\rho'' + r\rho' + (k^2r^2 - m^2)\rho = p(p-1)r^p + pr^p + \underbrace{k^2 r^{p+2}}_{\substack{\text{negligible} \\ \text{for small } r \\ \text{compared to } r^p}} - m^2 r^p$$

$$\approx r^p\left[p(p-1) + p - m^2\right]$$

$$= r^p\left(p^2 - m^2\right)$$

$\Rightarrow p = \pm m \Rightarrow$ two possible solutions:

$$\boxed{\begin{array}{ll} \text{1st kind: } & J_m(kr) \sim r^m \\ \text{Bessel func of 2nd kind: } & Y_m(kr) \sim r^{-m} \end{array}}$$

for small kr

[$m=0$ case is trickier: $Y_0(kr) \sim \log(r)$]

* Here, Y_m is not an allowed eigenfunction since we require finite solutions at $r \to 0$

$\Rightarrow$ eigenfunctions are:

$J_m(k_{mn}r)\cos(m\theta)$ and $J_m(k_{mn}r)\sin(m\theta)$ } "degenerate": 2 indep. u for each λ

for $\lambda_{mn} = -k_{mn}^2$, $k_{mn} = \frac{\xi_{mn}}{R}$