9. Fermionic integrals

9.1. **Bosons and fermions.** In physics there exist two kinds of particles — bosons and fermions. So far we have dealt with bosons only, but many important particles are fermions: e.g., electron, proton, etc. Thus it is important to adapt our techniques to the fermionic case.

In quantum theory, the difference between bosons and fermions is as follows: if the space of states of a single particle is $\mathcal{H}$ then the space of states of the system of k such particles is $S^k\mathcal{H}$ for bosons and $\Lambda^k\mathcal{H}$ for fermions. In classical theory, this means that the space of states of a bosonic particle is a usual real vector space (or more generally a manifold), while for a fermionic particle it is an *odd* vector space. Mathematically "odd" means that the ring of smooth functions on this space (i.e. the ring of classical observables) is an *exterior* algebra (unlike the case of a usual, *even* space, for which the ring of polynomial functions is a *symmetric* algebra).

More generally, one may consider systems of classical particles or fields some of which are bosonic and some fermionic. In this case, the space of states will be a supervector space, i.e. the direct sum of an even and an odd space (or, more generally, a supermanifold — a notion we will define below).

When such a theory is quantized using the path integral approach, one has to integrate functions over supermanifolds. Thus, we should learn to integrate over supermanifolds and then generalize to this case our Feynman diagram techniques. This is what we do in this section.

9.2. **Supervector spaces.** Let k be a field of characteristic zero. A supervector space (or shortly, superspace) over k is just a $\mathbb{Z}/2$-graded vector space: $V = V_0 \oplus V_1$. If $V_0 = k^n$ and $V_1 = k^m$ then V is denoted by $k^{n|m}$. The notions of a homomorphism, direct sum, tensor product, dual space for supervector spaces are defined in the same way as for $\mathbb{Z}/2$-graded vector spaces. In other words, the tensor category of supervector spaces is the same as that of $\mathbb{Z}/2$-graded vector spaces.

However, the notions of a supervector space and a $\mathbb{Z}/2$-graded vector space are *not* the same. The difference is as follows. The category of vector (and hence $\mathbb{Z}/2$-graded vector) spaces has an additional *symmetry* structure, which is the standard isomorphism $V\otimes W \to W\otimes V$ (given by $v\otimes w \to w\otimes v$). This isomorphism allows one to define symmetric powers S^mV, exterior powers Λ^mV, etc. For supervector spaces, there is also a symmetry $V \otimes W \to W \otimes V$, but it is defined differently. Namely, $v \otimes w$ goes to $(-1)^{mn} w \otimes v$, $v \in V_m, w \in V_n$ $(m, n \in \{0, 1\})$. In other words, it is the same as usual except that if v, w are odd then $v \otimes w \to -w \otimes v$. As a result, we can define the superspaces S^mV and Λ^mV for a superspace V, but they are not the same as the symmetric and exterior powers in the usual sense. For example, if V is purely odd ($V = V_1$), then S^mV is the exterior m-th power of V, and Λ^mV is the m-th symmetric power of V (purely even for even m and purely odd for odd m).

For a superspace V, let ΠV be the same space with opposite parity, i.e. $(\Pi V)_i = V_{1-i}$, $i = 0, 1$. With this notation, the equalities explained in the previous paragraph can be written as: $S^mV = \Pi^m(\Lambda^m\Pi V)$, $\Lambda^mV = \Pi^m(S^m\Pi V)$.

Let $V = V_0 \oplus V_1$ be a finite dimensional superspace. Define the algebra of polynomial functions on V, $\mathcal{O}(V)$, to be the algebra SV^* (where symmetric powers are taken in the super sense). Thus, $\mathcal{O}(V) = SV_0^* \otimes \Lambda V_1^*$, where V_0 and V_1 are regarded as usual spaces. More explicitly, if $x_1, \ldots, x_n$ are linear coordinates on V_0, and $\xi_1, \ldots, \xi_m$ are linear coordinates on V_1, then $\mathcal{O}(V) = k[x_1, \ldots, x_n, \xi_1, \ldots, \xi_m]$, with defining relations

$$x_ix_j = x_jx_i, x_i\xi_r = \xi_rx_i, \xi_r\xi_s = -\xi_s\xi_r$$

Note that this algebra is itself a (generally, infinite dimensional) supervector space, and is commutative in the supersense. Also, if V, W are two superspaces, then $\mathcal{O}(V \oplus W) = \mathcal{O}(V) \otimes \mathcal{O}(W)$, where the tensor product of algebras is understood in the supersense (i.e. $(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)}(ac \otimes bd)$, where $p(x)$ is the parity of x).

9.3. **Supermanifolds.** Now assume that $k = \mathbb{R}$. Then by analogy with the above for any supervector space V we can define the algebra of smooth functions, $C^\infty(V) := C^\infty(V_0) \otimes \Lambda V_1^*$. In fact, this is a special case of the following more general setting.

Definition 9.1. A supermanifold M is a usual manifold M_0 with a sheaf C_M^∞ of $\mathbb{Z}/2\mathbb{Z}$ graded algebras (called the structure sheaf), which is locally isomorphic to $C_{M_0}^\infty \otimes \Lambda(\xi_1, \ldots, \xi_m)$.

The manifold M_0 is called the reduced manifold of M. The dimension of M is the pair of integers $\dim M_0|m$.

For example, a supervector space V is a supermanifold of dimension $\dim V_0|\dim V_1$. Another (more general) example of a supermanifold is a superdomain $U := U_0 \times V_1$, i.e. a domain $U_0 \subset V_0$ together with the sheaf $C_U^\infty \otimes \Lambda V_1^*$. Moreover, the definition of a supermanifold implies that any supermanifold is "locally isomorphic" to a superdomain.

Let M be a supermanifold. An open set U in M is the supermanifold $(U_0,\, C_M^\infty|_{U_0})$, where U_0 is an open subset in M_0.

By the definition, supermanifolds form a category $\mathcal{SMAN}$. Let us describe explicitly morphisms in this category, i.e. maps $F : M \to N$ between supermanifolds M and N. By the definition, it suffices to assume that M, N are superdomains, with global coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_m$, and $y_1, \ldots, y_p, \eta_1, \ldots, \eta_q$, respectively (here x_i, y_i are even variables, and ξ_i, η_i are odd variables). Then the map F is defined by the formulas:

$$y_i = f_{0,i}(x_1, \ldots, x_n) + f_{2,i}^{j_1 j_2}(x_1, \ldots, x_n)\xi_{j_1}\xi_{j_2} + \cdots\,,$$

$$\eta_i = a_{1,i}^{j}(x_1, \ldots, x_n)\xi_j + a_{3,i}^{j_1 j_2 j_3}(x_1, \ldots, x_n)\xi_{j_1}\xi_{j_2}\xi_{j_3} + \cdots\,,$$

where $f_{0,i}, f_{2,i}^{j_1 j_2}, a_{1,i}^{j}, a_{3,i}^{j_1,j_2,j_3}, \ldots$ are usual smooth functions, and we assume summation over repeated indices. These formulas, determine F completely, since for any $g \in C^\infty(N)$ one can find $g \circ F \in C^\infty(M)$ by Taylor's formula. For example, if $M = N = \mathbb{R}^{1|2}$ and $F(x, \xi_1, \xi_2) = (x + \xi_1\xi_2, \xi_1, \xi_2)$, and if $g = g(x)$, then $g \circ F(x, \xi_1, \xi_2) = g(x + \xi_1\xi_2) = g(x) + g'(x)\xi_1\xi_2$.

Remark. For this reason, one may consider only C^∞ (and not C^r) functions on supermanifolds. Indeed, if for example $g(x)$ is a C^r function of one variable which is not differentiable $r+1$ times, then the expression $g(x + \sum_{i=1}^{r+1} \xi_{2i-1}\xi_{2i})$ will not be defined, because the coefficient of $\xi_1 \cdots \xi_{2r+2}$ in this expression should be $g^{(r+1)}(x)$, but this derivative does not exist.

9.4. **Supermanifolds and vector bundles.** Let M_0 be a manifold, and E be a vector bundle on M_0. Then we can define the supermanifold $M := \mathrm{Tot}(\Pi E)$, the total space of E with changed parity. Namely, the reduced manifold of M is M_0, and the structure sheaf C_M^∞ is the sheaf of sections of ΛE^*. This defines a functor $S : \mathcal{BUN} \to \mathcal{SMAN}$, from the category of manifolds with vector bundles to the category of supermanifolds. We also have a functor S_* in the opposite direction: namely, $S_*(M)$ is the manifold M_0 with the vector bundle $(R/R^2)^*$, where R is the nilpotent radical of C_M^∞.

The following proposition (whose proof we leave as an exercise) gives a classification of supermanifolds.

Proposition 9.2. *$S_*S = Id$, and $SS_* = Id$ on isomorphism classes of objects.*

The usefulness of this proposition is limited by the fact that, as one can see from the above description of maps between supermanifolds, SS_* is *not* the identity on morphisms (e.g. it maps the automorphism $x \to x + \xi_1\xi_2$ of $\mathbb{R}^{1|2}$ to Id), and hence, S is not an equivalence of categories. In fact, the category of supermanifolds is not equivalent to the category of manifolds with vector bundles (namely, the category of supermanifolds "has more morphisms").

Remark. The relationship between these two categories is quite similar to the relationship between the categories of (finite dimensional) filtered and graded vector spaces, respectively (namely, for them we also have functors S, S_* with the same properties – check it!). Therefore in supergeometry, it is better to avoid using realizations of supermanifolds as $S(M_0, E)$, similarly to how in linear algebra it is better to avoid choosing a grading on a filtered space.

9.5. **Integration on superdomains.** We would now like to develop integration theory on supermanifolds. Before doing so, let us recall how it is done for usual manifolds. In this case, one proceeds as follows.

1. Define integration of compactly supported (say, smooth) functions on a domain in $\mathbb{R}^n$.

2. Find the transformation formula for the integral under change of coordinates (i.e. discover the factor $|J|$, where J is the Jacobian).

3. Define a density on a manifold to be a quantity which is locally the same as a function, but multiplies by $|J|$ under coordinate change (unlike true functions, which don't multiply by anything).

Then define integral of compactly supported functions on the manifold using partitions of unity. The independence of the integral on the choices is guaranteed by the change of variable formula and the definition of a density.

We will now realize this program for supermanifolds. We start with defining integration over superdomains.

Let $V = V_0 \oplus V_1$ be a supervector space. The *Berezinian* of V is the line $\Lambda^{\mathrm{top}} V_0 \otimes \Lambda^{\mathrm{top}} V_1^*$. Suppose that V is equipped with a nonzero element dv of the Berezinian (called a supervolume element).

Let U_0 be an open set in V_0, and $f \in C^\infty(U) \otimes \Lambda V_1^*$ be a compactly supported smooth function on the superdomain $U := U_0 \times V_1$ (i.e. $f = \sum f_i \otimes \omega_i$, $f_i \in C^\infty(U)$, $\omega_i \in \Lambda V_1^*$, and f_i are compactly supported). Let dv_0, dv_1 be volume forms on V_0, V_1 such that $dv = dv_0/dv_1$.

Definition 9.3. The integral $\int_U f(v)dv$ is $\int_{U_0} (f(v), (dv_1)^{-1}) dv_0$.

It is clear that this quantity depends only on dv and not on dv_0 and dv_1 separately.

Thus, $\int f(v)dv$ is defined as the integral of the suitably normalized top coefficient of f (expanded with respect to some homogeneous basis of ΛV_1^*). To write it in coordinates, let $\xi_1, \ldots, \xi_m$ be a linear system of coordinates on V such that $dv = \frac{dx_1 \cdots dx_n}{d\xi_1 \cdots d\xi_m}$ (such coordinate systems will be called unimodular with respect to dv). Then $\int f(v)dv$ equals $\int f_{\mathrm{top}}(x_1, \ldots, x_n) dx_1 \cdots dx_n$, where f_{top} is the coefficient of $\xi_1 \cdots \xi_n$ in the expansion of f.

9.6. **The Berezinian of a matrix.** Now we generalize to the supercase the definition of determinant (since we need to generalize Jacobian, which is a determinant).

Let R be a supercommutative ring. Fix two nonnegative integers m, n. Let A be a $n+m$ by $n+m$ matrix over R. Split A in the blocks $A_{11}, A_{12}, A_{21}, A_{22}$ so that A_{11} is n by n, and A_{22} is m by m. Assume that the matrices A_{11}, A_{22} have even elements, while A_{21} and A_{12} have odd elements. Assume also that A_{22} is invertible.

Definition 9.4. The Berezinian of A is the element

$$\mathrm{Ber}(A) := \frac{\det(A_{11} - A_{12} A_{22}^{-1} A_{21})}{\det(A_{22})} \in R$$

(where the determinant of the empty matrix is agreed to be 1; so for $m = 0$ one has $\mathrm{Ber} A = \det A$, and for $n = 0$ one has $\mathrm{Ber} A = (\det A)^{-1}$).

Remark. Recall for comparison that if A is purely even then

$$\det(A) := \det(A_{11} - A_{12} A_{22}^{-1} A_{21}) \det(A_{22}).$$

The Berezinian has the following simpler description. Any matrix A as above admits a unique factorization $A = A_+ A_0 A_-$, where A_+, A_0, A_- are as above, and in addition A_+, A_- are block upper (respectively, lower) triangular with 1 on the diagonal, while A_0 is block diagonal. Then $\mathrm{Ber}(A) = \det((A_0)_{11}) / \det((A_0)_{22})$.

Proposition 9.5. *If A, B be matrices as above, then* $\mathrm{Ber}(AB) = \mathrm{Ber}(A)\mathrm{Ber}(B)$.

Proof. From the definition using triangular factorization, it is clear that it suffices to consider the case $A = A_-$, $B = B_+$. Let $X = (A_-)_{21}$, $Y = (B_+)_{12}$ (matrices with odd elements). Then the required identity is

$$\det(1 - Y(1 + XY)^{-1} X) = \det(1 + XY).$$

To prove this relation, let us take the logarithm of both sides and expand using Taylor's formula. Then the left hand side gives

$$-\sum_{k \geq 1} \mathrm{Tr}(Y(1 + XY)^k (XY)^{k-1} X)/k$$

Using the cyclic property of the trace, we transform this to

$$\sum_{k \geq 1} \mathrm{Tr}((1 + XY)^k (XY)^k)/k$$

(the minus disappears since X, Y have odd elements). Summing the series, we find that the last expression equals

$$-\mathrm{Tr}\ln(1-(1+XY)^{-1}XY) = \mathrm{Tr}\ln(1+XY),$$

as desired. $\square$

The additive analog of Berezinian is supertrace. Namely, for A as above, $\mathrm{sTr}A = \mathrm{Tr}A_{11} - \mathrm{Tr}A_{22}$. It is the correct superanalogue of the usual trace, as it satisfies the equation $\mathrm{sTr}(AB) = \mathrm{sTr}(BA)$ (while the usual trace does not). The connection between the supertrace and the Berezinian is given by the formula

$$\mathrm{Ber}(e^A) = e^{\mathrm{sTr(A)}}.$$

Exercise. Prove this formula.

9.7. **Berezin's change of variable formula.** Let V be a vector space, $f \in \Lambda V^*$, $v \in V$. Denote by $\frac{\partial f}{\partial v}$ the result of contraction of f with v.

Let U, U' be superdomains, and $F : U \to U'$ be a morphism. As explained above, given linear coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_m$ on U and $y_1, \dots, y_p, \eta_1, \dots, \eta_q$ on U', we can describe f by expressing y_i and η_i as functions of x_j and ξ_j. Define the Berezin matrix of F, $A := DF(x, \xi)$ by the formulas:

$$A_{11} = (\frac{\partial y_i}{\partial x_j}),\ A_{12} = (\frac{\partial y_i}{\partial \xi_j}),\ A_{21} = (\frac{\partial \eta_i}{\partial x_j}),\ A_{22} = (\frac{\partial \eta_i}{\partial \xi_j}).$$

Clearly, this is a superanalog of the Jacobi matrix.

The main theorem of supercalculus is the following theorem.

Theorem 9.6. *(Berezin) Let g be a smooth function with compact support on U', and $F : U \to U'$ be an isomorphism. Let dv, dv' be supervolume elements on U, U'. Then*

$$\int_{U'} g(v')dv' = \int_U g(F(v))|\mathrm{Ber}DF(v)|dv,$$

where the Berezinian is computed with respect to unimodular coordinate systems.

Remark. If $f(\xi) = a$+terms containing ξ_j then by definition $|f(\xi)| := f(\xi)$ is $a > 0$ and $-f(\xi)$ if $a < 0$.

Proof. The chain rule of the usual calculus extends verbatim to supercalculus. Also, we have shown that $\mathrm{Ber}(AB) = \mathrm{Ber}(A)\mathrm{BerB}$. Therefore, if we know the statement of the theorem for two isomorphisms $F_1 : U_2 \to U_1$ and $F_2 : U_3 \to U_2$, then we know it for the composition $F_1 \circ F_2$.

Let $F(x_1, \dots, x_n, \xi_1, \dots, \xi_m) = (x'_1, \dots, x'_n, \xi'_1, \dots, \xi'_m)$. From what we just explained it follows that it suffices to consider the following cases.

1. x'_i depend only on x_j, $j = 1, \dots, n$, and $\xi'_i = \xi_i$.
2. $x'_i = x_i + z_i$, where z_i lie in the ideal generated by ξ_j, and $\xi'_i = \xi_i$.
3. $x'_i = x_i$.

Indeed, it is clear that any isomorphism F is a composition of isomorphisms of the type $1, 2, 3$.

In case 1, the statement of the theorem follows from the usual change of variable formula. Thus it suffices to consider cases 2 and 3.

In case 2, it is sufficient to consider the case when only one coordinate is changed by F, i.e. $x'_1 = x_1 + z$, and $x'_i = x_i$ for $i \geq 2$. In this case we have to show that the integral of

$$g(x_1 + z, x_2, \dots, x_n, \xi)(1 + \frac{\partial z}{\partial x_1}) - g(x, \xi)$$

is zero. But this follows easily upon expansion in powers of z, since all the terms are manifestly total derivatives with respect to x_1.

In case 3, we can also assume $\xi'_i = \xi_i$, $i \geq 2$, and a similar (actually, even simpler) argument proves the result. $\square$

9.8. **Integration on supermanifolds.** Now we will define densities on supermanifolds. Let M be a supermanifold, and $\{U_\alpha\}$ be an open cover of M together with isomorphisms $f_\alpha : U_\alpha \to U'_\alpha$, where U'_α is a superdomain in $\mathbb{R}^{n|m}$. Let $g_{\alpha\beta} : f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ be the transition map $f_\alpha f_\beta^{-1}$. Then a density s on M is a choice of an element $s_\alpha \in C_M^\infty(U_\alpha)$ for each α, such that on $U_\alpha \cap U_\beta$ one has $s_\beta(z) = s_\alpha(z)|\mathrm{Ber}(g_{\alpha\beta})(f_\beta(z))|$.

Remark. It is clear that a density on M is a global section of a certain sheaf on M, called the sheaf of densities.

Now, for any (compactly supported) density ω on M, the integral $\int_M \omega$ is well defined. Namely, it is defined as is usual calculus: one uses partition of unity ϕ_α such that $\mathrm{Supp}\phi_\alpha \subset (U_\alpha)_0$ are compact subsets, and sets $\int_M \omega := \sum_\alpha \int_M \phi_\alpha \omega$ (where the summands can be defined using f_α). Berezin's theorem guarantees then that the final answer will be independent on the choices made.

9.9. **Gaussian integrals in an odd space.** Now let us generalize to the odd case the theory of Gaussian integrals, which was, in the even case, the basis for the path integral approach to quantum mechanics and field theory.

Recall first the notion of Pfaffian. Let A be a skew-symmetric matrix of even size. Then the determinant of A is the square of a polynomial in the entries of A. This polynomial is determined by this condition up to sign. The sign is usually fixed by requiring that the polynomial should be 1 for the direct sum of matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. With this convention, this polynomial is called the Pfaffian of A and denoted PfA. The Pfaffian obviously has the property $\mathrm{Pf}(X^TAX) = \mathrm{Pf}(A)\det(X)$ for any matrix X.

Let now V be an $2m$-dimensional vector space with a volume element dv, and B a skew-symmetric bilinear form on V. We define the Pfaffian PfB of B to be the Pfaffian of the matrix of B in any unimodular basis by the above transfoamtion formula, it does not depend on the choice of the basis). It is easy to see (by reducing B to canonical form) that

$$\frac{\wedge^m B}{m!} = \mathrm{Pf}(B)dv.$$

In terms of matrices, this translates into the following (well known) formula for the Pfaffian of a skew symmetric matrix of size $2m$:

$$\mathrm{Pf}(A) = \sum_{\sigma\in\Pi_m} \varepsilon_\sigma \prod_{i\in\{1,\dots,2m\}, i<\sigma(i)} a_{i\sigma(i)},$$

where Π_m is the set of pairings of $\{1, \dots, 2m\}$, and ε_σ is the sign of the permutation sending $1, \dots, 2m$ to $i_1, \sigma(i_1), \dots, i_m, \sigma(i_m)$ (where $i_r < \sigma(i_r)$ for all r). For example, for $m = 2$ (i.e. a 4 by 4 matrix),

$$\mathrm{Pf}(A) = a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24}.$$

Now consider an odd vector space V of dimension $2m$ with a volume element $d\xi$. Let B be a symmetric bilinear form on V (i.e. a skewsymmetric form on ΠV). Let $\xi_1, \dots, \xi_{2m}$ be unimodular linear coordinates on V (i.e. $d\xi = d\xi_1 \wedge \cdots \wedge d\xi_m$). Then if $\xi = (\xi_1, \dots, \xi_n)$ then $B(\xi, \xi) = \sum_{i,j} b_{ij}\xi_i\xi_j$, where b_{ij} is a skewsymmetric matrix.

Proposition 9.7.

$$\int_V e^{\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1} = \mathrm{Pf}(B).$$

Proof. The integral equals $\frac{\wedge^m B}{m!d\xi}$, which is $\mathrm{Pf}(B)$. □

Example. Let V is a finite dimensional odd vector space, and $Y = V \oplus V^*$. The space Y has a canonical volume element $dvdv^*$, defined as follows: if $e_1, \dots, e_m$ be a basis of V and $e_1^*, \dots, e_n^*$ is the dual basis of V^* then $dvdv^* = e_1 \wedge e_1^* \cdots \wedge e_n \wedge e_n^*$. Let $dy = (dvdv^*)^{-1}$ be the corresponding supervolume element.

Let $A : V \to V$ be a linear operator. Then we can define an even smooth function S on the odd space Y as follows: $S(v, v^*) = (Av, v^*)$. More explicitly, if ξ_i be coordinates on V corresponding to the basis e_i, and η_i the dual system of coordinates on V^*, then

$$S(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_m) = \sum a_{ij}\xi_j\eta_i,$$

where (a_{ij}) is the matrix of A in the basis e_i.

Proposition 9.8.

$$\int_Y e^S dy = \det A$$

Proof. We have $S(v, v_*) = \frac{1}{2}B((v, v_*), (v, v_*))$, where B is the skew form on ΠY, which is given by the formula $B((v, v^*), (w, w^*)) = (Av, w^*) - (Aw, v^*)$. It is easy to see that $\mathrm{Pf}(B) = \det(A)$, so Proposition 9.8 follows from Proposition 9.7.

Another proof can be obtained by direct evaluation of the top coefficient. $\square$

9.10. **The Wick formula in the odd case.** Let V be a $2m$-dimensional odd space with a volume form $d\xi$, and $B \in S^2V$ a nondegenerate form (symmetric in the supersense and antisymmetric in the usual sense). Let $\lambda_1, \dots, \lambda_n$ be linear functions on V (regarded as the usual space). Then $\lambda_1, \dots, \lambda_n$ can be regarded as odd smooth functions on the superspace V.

Theorem 9.9.

$$\int_V \lambda_1(\xi) \cdots \lambda_n(\xi) e^{-\frac{1}{2}B(\xi,\xi)} (d\xi)^{-1} = \mathrm{Pf}(-B)\mathrm{Pf}(B^{-1}(\lambda_i, \lambda_j)).$$

(By definition, this is zero if n is odd). In other words, we have:

$$\int_V \lambda_1(\xi) \cdots \lambda_n(\xi) e^{-\frac{1}{2}B(\xi,\xi)} (d\xi)^{-1} = \mathrm{Pf}(-B) \sum_{\sigma \in \Pi_m} \varepsilon_\sigma \prod_{i \in \{1,\dots,2m\}, i < \sigma(i)} (B^{-1}(\lambda_i, \lambda_{\sigma(i)})).$$

Proof. We prove the second formula. Choose a basis e_i of V with respect to which the form B is standard: $B(e_j, e_l) = 1$ if $j = 2i - 1, l = 2i$, and $B(e_j, e_l) = 0$ for other pairs $j < l$. Since both sides of the formula are polylinear with respect to $\lambda_1, \dots, \lambda_n$, it suffices to check it if $\lambda_1 = e^*_{i_1}, \dots, \lambda_n = e^*_{i_n}$. This is easily done by direct computation (in the sum on the right hand side, only one term may be nonzero). $\square$