# 18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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We'll start by finishing the Lindstrom-Gessel-Viennot lemma proof. Recall the statement:

**Theorem 1**

Given a directed acyclic digraph $G$ with some vertices $A_1, \cdots A_k, B_1, \cdots, B_k$, the (weighted with a $\pm$) number of ways to connect each $A_i$ to $B_i$ for $1 \le i \le k$ is the determinant

$$\sum_{w \in S_n} (-1)^{\ell(w)} N(A_1, \cdots, A_k, B_{w(1)}, \cdots, B_{w(k)}) = \det C$$

where $C_{ij}$ is the number of ways to connect $A_i$ to $B_j$ with no other restrictions.

*Proof.* First, expand out $\det C$ as we did last class - see the notes there for details.

The idea is to construct a sign-reversing involution $\sigma$ on all bad collections of paths: basically, if we have two collections of paths with intersection, we find the "first intersection" $(P_i, P_j, X)$, where the paths $P_i$ and $P_j$ intersect at a point $X$, and then swap $P_i$ and $P_j$ after point $X$.

We can't just take the lexicographic minimum $(i, j)$, because the minimum is not preserved - there might be a new lexicographic minimum! Instead, take the minimal index $i$ such that $P_i$ has an intersection with another path. Let the first intersection along that path (with another $P_j$) be point $X$: however, there can still be a bunch of other paths through $X$. Among all of those, **find $P_j$ with the smallest index $j$.**

Then after we apply $\sigma$, we'll still pick $P_j$ and $P_i$ at point $X$, and indeed $\sigma$ reverses the sign of our permutation! Thus all such terms do cancel out, as desired. $\square$

This lemma has a weighted version as well! Let's now say that every edge $e$ in $G$ is assigned some positive weight $x_e > 0$, and define

$$N(A_1, \cdots, A_k, B_1, \cdots, B_k) = \sum_{\substack{\text{noncrossing paths} \\ P_1, P_2, \cdots, P_k}} \prod_i \prod_{e \in P_i} x_e.$$

**Theorem 2**

Then this quantity

$$\sum_{w \in S_n} (-1)^{\ell(w)} N(A_1, \cdots, A_k, B_{w(1)}, \cdots, B_{w(k)})$$

is again the determinant of $C$, where $c_{ij} = N(A_i, B_j)$.

The same proof works: we just need to see that the sign-reversing involution also preserves the weights, because the multiset of all edges is preserved!

**Fact 3**

Also, we can make the same assumption that we had before: if the only way to connect the $A_i$s to the $B_j$s is $A_1 \to B_1, A_2 \to B_2, \cdots$, then the left side is just $N(A_1, \cdots, A_k, B_1, \cdots, B_k)$.

So how can we make sure that this assumption is indeed true, so that our theorem is a lot simpler?

**Example 4**

Let's say $G$ is a **planar graph** that can be drawn in a square so that all the sources $A_1, \cdots, A_k$ are on the left side, and all the sinks $B_1, \cdots, B_k$ are on the right side.

Then we can't connect $A_1$ to anything beside $B_1$, and so on! In particular, our matrix $C$ will have determinant

$$\det C = N(A_1, \cdots, A_n, B_1, \cdots, B_n) \geq 0,$$

because it's just some weighted sum of positive-valued paths. We can actually say a lot more about matrices of this form!

**Definition 5**

A matrix $C$ with real entries is **totally positive** (respectively, **totally nonnegative**) if all minors (determinants of square submatrices) are greater than 0 (respectively, nonnegative).

This is much stronger than having a matrix that is **positive definite**, which only requires that principal minors are positive!

**Example 6**

For a 3 by 3 totally positive matrix, we know that all entries are positive, and any square submatrix also has positive determinant! This is a lot of conditions.

Well, note that in our planar graph, we can always remove any $k$ entries from the left and any $k$ entries from the right, and we still have only one unique way to choose our path endpoints! **So all such matrices $C$ resulting from planar graphs $G$ this way are totally nonnegative.**

**Fact 7**

It turns out that a matrix $C$ is totally nonnegative **if and only if** it can be represnted by a planar graph in this way!

So somehow there's a deep connection between these two ideas of planarity and total nonnegativity. How can we generally get any $m$ by $n$ totally positive matrix?

Start by drawing an $m$ by $n$ grid, so that all vertical edges are directed up and all horizontal edges are directed to the right. We'll only put weights on the horizontal edges to avoid redundancy: now label them $x_{11}, x_{12}, \cdots$ on the top row, $x_{21}, x_{22}, \cdots$ on the next row, and so on.

We'll put our $A_i$s from left to right on the bottom row, and we'll put our $B_i$s from top to bottom on the right column (this is equivalent to putting the $A$s on the left by deformation). Now we have some matrix $C$ such that $c_{ij} = N(A_i, B_j)$.

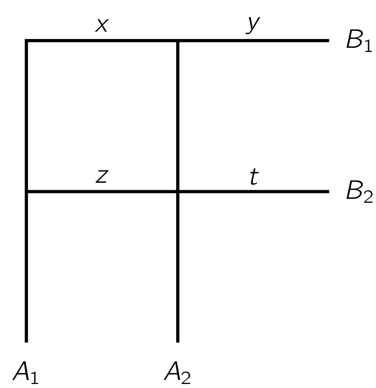
**Theorem 8**

Then any totally positive matrix can be represented using some $x_{ij}$s.

**Example 9**

Take $m = n = 2$.

Here's what our graph looks like:




In this case, all entries are positive (since weights are positive), and the determinant is also positive (in particular, the only way to connect $A_i$s to $B_j$s is by using edges $x, y, t$, so the determinant is $xyt$).

**Corollary 10**

Thus, the space of all totally positive $m \times n$ matrices is isomorphic to $(\mathbb{R}_{>0})^{mn}$.

Let's talk about another application of the Lindstrom lemma: we're going to go back to Young diagram-like objects by thinking about plane partitions!

Consider an $m \times n$ rectangle broken up into 1 by 1 grid squares. We again put numbers into our grid squares, but our entries are now weakly decreasing, and we're allowed to repeat numbers. Here's an example:

| | | | | |
|---|---|---|---|---|
| 7 | 7 | 6 | 6 | 4 |
| 7 | 6 | 5 | 4 | 3 |
| 5 | 5 | 3 | 3 | 3 |
| 3 | 3 | 3 | 2 | 1 |
| 2 | 2 | 2 | 1 | 1 |

Let's say that all entries are in $\{1, k\}$. Is it possible to calculate the number of such plane partitions? Here's the complete list for $m = n = k = 2$:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & 2 \\ \hline\end{array}$$

**Example 11**

For $k = 2$, the number of such plane partitions is $\binom{n+m}{n}$: this is because we just need to find the path that separates the 1s and 2s from each other.

Can we extend this logic in general? We can again draw paths $P_k$ that separate entries $> k$ from those $\leq k$ for all $k$, and this creates $(k-1)$ **weakly noncrossing paths** in an $m \times n$ grid: they can intersect but not cross over each other.

But now just slightly perturb each path: have $P_{k-1}$ starting at $A_{k-1} = (0, 0)$ and ending at $B_{k-1} = (m, n)$, $P_{k-2}$ starting at $(1, -1)$ and ending at $(m+1, n-1)$, and so on. Now the paths are non-crossing, and we can apply the Lindstrom lemma: we have a $(k-1)$ by $(k-1)$ matrix with entries $c_{ij} = \binom{m+n}{m+i-j}$ (which is just the number of ways to get from $A_i$ to $B_j$. There's actually an explicit product formula, but we'll skip over it for now!

Well, we can think of our plane partitions as three-dimensional Young diagrams: make each number $c$ into a tower of $c$ cubes stacked on top of each other! In general, if we rotate our picture, this can be bijected into **rhombus tilings** of hexagons with pairs of opposites of length $m, n, k$. This tells us that the condition is actually symmetric on $m, n, k$, and we'll look more at the explicit formula at some point.

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