18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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Let's go back to the unimodality of the Gaussian coefficients - the easiest proof is to use linear algebra instead of a combinatorial proof. Remember that we consider

$$\begin{bmatrix} k+l \\ k \end{bmatrix}_q = a_) + a_1 q + \cdots + a_{kl} q^{kl},$$

and we want to show the coefficients are increasing and then decreasing.

The idea is to consider V_n, the linear space of formal linear combinations of Young diagrams λ in a k by l rectangle with n squares. The dimension is just a_n, the number of possible Young diagrams, and our goal is to show that $a_i \le a_{i+1}$ for all $i < \frac{kl}{2}$: by symmetry of the coefficients, we get the result.

Consider a **weighted up operator** $U_n : V_n \to V_{n+1}$ which sends

$$\lambda \to \sum_{\substack{\mu = \lambda \cup \{x\} \\ \mu \gtrdot \lambda}} \sqrt{w(x)} \mu,$$

where w is a weight function sending boxes of $k \times l$ rectangles to positive reals. Similarly, we consider the **weighted down operator** $D_n : V_{n+1} \to V_n$, sending

$$\lambda \to \sum_{\substack{\mu = \lambda \setminus \{x\} \\ \mu \lessdot \lambda}} \sqrt{w(x)} \mu.$$

Define our commutator

$$H_n = D_n U_n - U_{n-1} D_{n-1};$$

notice that this takes any element in V_n to another element in V_n. We can represent U_n with an a_{n+1} by a_n matrix, and we can represent D_n by its transpose: U_n^T.

Fact 187

They are transposes, because all nonzero entries in U_n have $\sqrt{x}$ in the entry (a, b), where a and b differ by the box x, and entries in D_n have $\sqrt{x}$ in the entry (b, a).

Claim 187.1. H_n is a diagonal matrix with entries

$$(H_n)_{\lambda,\lambda} = \sum_{x \in \text{Add}(\lambda)} w(x) - \sum_{y \in \text{Remove}(\lambda)} w(y).$$

Why is this? The off-diagonal entries mean we start with a Young diagram, add a box, and remove a different box: this is equivalent to first removing the other box and then add it, so those always cancel out. Meanwhile, the diagonal entries get a $\sqrt{w(x)}^2 = w(x)$ contribution.

Here's a diagram: the As are part of Add(λ), while the Rs are part of Remove(λ).

		R	A
	A		
R			

So let's assume we can find a weight function w so that the matrix H_n has positive diagonal entries: thus, the eigenvalues are all positive. Then

$$D_n U_n = U_{n-1} D_{n-1} + H_n = U_{n-1} U_{n-1}^T + H_n.$$

$U_{n-1} U_{n-1}^T$ is positive semi-definite (since for any matrix A, xAA^Tx is the square of the standard dot product of A^Tx with itself), and H_n is positive definite, so their sum is positive definite (this is a fact from linear algebra!) This means $D_n U_n$ has nonzero determinant, and therefore the rank of $D_n U_n$ is a_n.

Fact 188 (Other linear algebra fact)

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. Then the rank of AB is less than the minimum of m, n, k, since rank can't increase with products or be larger than the dimensions of the matrices!

So the rank of $D_n U_n$ is a_n, but D_n is an $a_n \times a_{n+1}$ matrix and U_n is $a_{n+1} \times a_n$. So $a_n \le \min(a_n, a_{n+1})$ and therefore $a_n \le a_{n+1}$, as desired! So as long as we can find a weight function, we are good.

Well, define $w : [k] \times [l] \to \mathbb{R}_{>0}$ as

$$w(i, j) = (i - j + l)(j - i + k), 1 \le i \le k, 1 \le j \le l.$$

Example 189

Here it is for $k = 3, l = 4$:

12	12	10	6
10	12	12	10
6	10	12	12

Note that all of these are of the form $n(7 - n)$, and it is larger closer to the centers.

We claim that this weight function works! Here's why:

Lemma 190

For all λ contained in a $k \times l$ box,

$$w_\lambda = \sum_{x \in \text{Add}(\lambda)} w(x) - \sum_{y \in \text{Remove}(\lambda)} w(y) = kl - 2|\lambda|.$$

So this is positive as long as $n = |\lambda| < \frac{kl}{2}$.

This will be an exercise! In the diagram above, the sum of the As is 18, while the sum of the Rs is 16.

Let's shift gears now and talk about partitions. Recall that $p(n)$ is the number of partitions of n, which is the number of Young diagrams with n boxes.

Theorem 191

The generating function

$$\sum_{n \geq 0} p(n) q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}.$$

Expanding this out, it is

$$(1 + q + q^2 + \cdots)(1 + q^2 + q^4 + \cdots)(1 + q^n + \cdots)$$

and it's okay to only go up to the first n terms, since all other terms will have a higher power! So we can always truncate this to a finite product with finite terms.

Proof. We know that

$$\begin{bmatrix} k+l \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times l} q^{|\lambda|}.$$

Take the limit as $k, l \to \infty$. Then we're summing over all Young diagrams, and just expand out the q-binomial coefficient! □

A better proof. We can encode our partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ by a different set of integers: let m_i be the number of times i appears in λ, that is, the number of js such that $\lambda_j = i$. Then we can encode the **multiplicities** n_i $\lambda = (1^{m_1} 2^{m_2} \cdots)$.

So now, a partition $(6, 4, 4, 3, 1, 1)$ is now encoded via $(1^2 2^0 3^1 4^2 5^0 6^1)$, which corresponds to picking out the q^2 term in $(1 + q + q^2 + \cdots)$, the 1 term in $(1 + q^2 + q^4 + \cdots)$, the q^3 term in $(1 + q^3 + q^6 + \cdots)$, and so on! More rigorously, the sum

$$\sum_{n \geq 0} p(n) q^n = \sum_{m_1, m_2, \cdots \geq 0} = q^{m_1 + 2m_2 + 3m_3 + \cdots}$$

can be factored as

$$\sum_{m_1} q^{m_1} \cdot \sum_{m_2} q^{2m_2} \cdot \sum_{m_3} q^{3m_3} \cdots$$

which is just

$$\frac{1}{1-q} \frac{1}{1-q^2} \cdots$$

as desired. □

There are also some special classes of partitions.

Definition 192

Define $p^{\text{odd}}(n)$ to be the number of partitions of n into odd parts: $\lambda = \lambda_1 + \lambda_2 + \cdots$, where all λ_i are odd.

This means $m_i = 0$ for all even i, and the generating function is

$$\frac{1}{1-q} \cdot \frac{1}{1-q^3} \cdot \frac{1}{1-q^5} \cdots .$$

Definition 193

Define $p^{\text{dist}}(n)$ to be the number of partitions of n into distinct parts: we have $\lambda = (\lambda_1 > \lambda_2 > \cdots)$.

This means $m_i \le 1$ for all i, so the generating function is

$$(1+q)(1+q^2)(1+q^3)\cdots .$$

Theorem 194 (Euler, 1748)

$p^{\text{odd}}(n) = p^{\text{dist}}(n)$.

For example, for $n = 5$, we can break it into odd parts as $5 = 3 + 1 + 1 = 1 + 1 + 1 + 1 + 1$, and we can break it into distinct parts as $5 = 4 + 1 = 3 + 2$.

Proof. Our goal is to check that the generating functions above are equal! Take the generating function for p^{dist}: it is

$$(1+q)(1+q^2)(1+q^3)\cdots = \frac{(1+q)(1-q)}{1-q} \cdot \frac{(1+q^2)(1-q^2)}{1-q^2} \cdot \frac{(1+q^3)(1-q^3)}{1-q^3} \cdots$$
$$= \frac{(1-q^2)(1-q^4)(1-q^6)\cdots}{(1-q)(1-q^2)(1-q^3)\cdots} = \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots},$$

which is the generating function for p^{odd} as desired. □

There's also a combinatorial proof! This is left as an exercise as well.

Theorem 195 (Euler's pentagonal number theorem, 1750)

We have

$$\frac{1}{\sum_{n \ge 0} p(n) q^n} = (1-q)(1-q^2)(1-q^3)\cdots = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2}.$$

This basically counts the number of partitions with even versus odd parts and finds their difference: apparently this is 0 for almost all values of n. Numbers of the form $m(3m-1)/2$ are called pentagonal numbers, because it's the number of dots in successive dilations of a pentagon!