Universal random structures in 2D

Introduction to 18.177, Fall 2015

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THE NUMBERS

24 lectures (after this one)

3 problem sets (to be assigned)

1 written project (research or expository) of about

5 pages per student, collaboration allowed

THE GOALS

Introduce some fundamental objects

Explain how they are related to each other

Explore some open problems

THE GOAL TODAY

Colloquium-style overview of

major objects and relationships

Overview

Prologue:

1. **Universality:** physics intuition, examples
2. **Discrete-continuum interplay:** scaling limits, discretizations
3. **Fractals and complex dynamics:** Julia sets, fractal dimensions, Mandelbrot, etc.

Part I: Cast of Characters: *What are the most fundamental 2D random objects?*

1. **Universal random trees:** Brownian motion, continuum random tree
2. **Universal random surfaces:** quantum gravity, planar maps, string theory, CFT
3. **Universal random paths:** walks, interfaces, Schramm-Loewner evolution, CFT
4. **Universal random growth:** Eden model, DLA, DBM

Part II: Drama: *How are the characters related to each other?*

1. **Welding random surfaces:** a calculus of random surfaces and SLE seams
2. **Mating random trees:** tree plus tree (conformally mated) equals surface plus path
3. **Random growth on random surfaces:** dendrites, dragons, surprising tractability
4. **Mating random trees produced by a snake:** metric spaces and the Brownian map
5. **Two "universal random surfaces" are the same:** Brownian map equals Liouville quantum gravity with parameter $\gamma = \sqrt{8/3}$ (a.k.a. "pure quantum gravity").

PROLOGUE:
UNIVERSALITY

Universality in physics (per Wikipedia)

In statistical mechanics, **universality** is the observation that there are properties for a large class of systems that are independent of the dynamical details of the system. Systems display universality in a **scaling limit**, when a large number of interacting parts come together. The modern meaning of the term was introduced by Leo Kadanoff in the 1960s, but a simpler version of the concept was already implicit in the van der Waals equation and in the earlier Landau theory of **phase transitions**, which did not incorporate scaling correctly. The term is slowly gaining a broader usage in several fields of **mathematics**, including **combinatorics and probability theory**, whenever the quantitative features of a structure (such as asymptotic behaviour) can be deduced from a few global parameters appearing in the definition, **without requiring knowledge of the details of the system**. The **renormalization group** explains universality. It classifies operators in a statistical field theory into relevant and irrelevant. Relevant operators are those responsible for perturbations to the free energy, the imaginary time Lagrangian, that will affect the continuum limit, and can be seen at long distances. Irrelevant operators are those that only change the short-distance details. The collection of scale-invariant statistical theories define the **universality classes**, and the finite-dimensional list of coefficients of relevant operators parametrize the near critical behavior.

Stories

- Physicists tell us that empirically many phenomena (such as phase transition exponents) are surprisingly similar from one material to another. Different microscopic setup, same "universality class."
- Sometimes simple toy mathematical models (percolation, Ising model, etc.) are said to belong to the same universality class as real world statistical physical systems.
- Mathematical physics game: try to identify the *very simplest* members of a given universality class and prove theorems about them. Maybe try tweaking the model and proving the theorems are still true.
- Example: Gaussian random variables (central limit theorem).
- Example: Brownian motion.
- Example: Brownian motion outer boundary (Mandelbrot 1982; Lawler, Schramm, Werner 2000).
- Example: percolation (Cardy 1992; Smirnov 2001).

PROLOGUE:

DISCRETE-CONTINUUM INTERPLAY

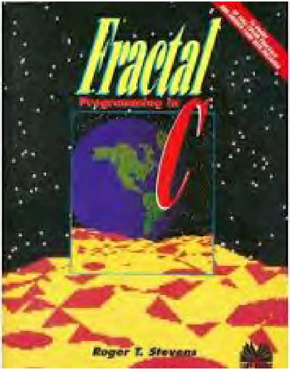
Discrete world vs. continuum world: more stories

- **Statistical physics:** argue that your (simple) continuum theory approximates your (not so simple) atomic model when the number of atoms is very large.
- **Particle physics:** argue that your (well defined) discrete lattice models approximate your (maybe complicated, maybe ill defined) continuum field theory when the lattice is very fine.
- **One mathematical goal:** develop continuum theories to help you understand scaling limits of beloved discrete models.
- **Another mathematical goal:** develop discrete approximations to help you understand beloved continuum theories (like Navier-Stokes and Yang-Mills).
- **Interplay** between the discrete and continuum is at the heart of many fields within physics and mathematics.
- **Mathematically rigorous** connections between discrete and continuum are sometimes hard to prove, which leads to....
- **Non-rigorous approach:** (common in physics) just *assume* you can pass from discrete to continuum and back whenever you need to. Then check whether end result seems to match experiments or simulations.
- **Conformal symmetry:** plays special role in 2D, following work by Belavin, Polyakov, Zamolodchikov and others in 1980's.

PROLOGUE:

NON-RANDOM FRACTALS FROM COMPLEX DYNAMICS

FRACTALS FROM COMPLEX DYNAMICS



Published 1989, by Roger T. Stevens

- Julia sets (Julia, 1918), popularized in 1980's
- Consider map $\phi(z) = z^2$.
- Maps $\mathbf{C} \setminus \overline{D}$ conformally to self (2 to 1) where D is unit disc. Repeated iteration takes points in $\mathbf{C} \setminus \overline{D}$ to ∞, leaves others bounded.
- If K is another compact set with connected hull, can construct a similar (2 to 1) conformal map ϕ_K from $\mathbf{C} \setminus \overline{K}$ to itself.
- Might expect more intricate sets K to yield more intricate maps. But suppose we take $\phi_K(z) = z^2 + c$ and let K be set of points remaining bounded under repeated iteration.
- K is a (filled) **Julia set**. Can "mate" Julia sets to form sphere (Douady 1983, Milnor 1994, see Arnaud Chéritat's animations).
- **Popular lexicon:** chaos theory, butterly effect, fractal, self-similar. What about *random* fractals, only self similar in law?

Part I:

CAST OF CHARACTERS

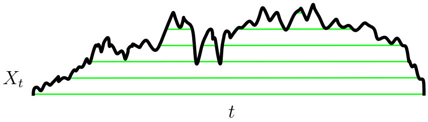
A Trees

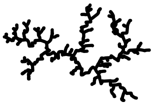
B Simple curves, non-simple curves, space-filling curves

C Surfaces

D Growth

RANDOM TREES

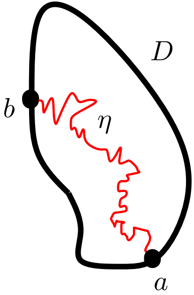




- This is the easiest "universal" random fractal to explain.
- Aldous (1993) constructs **continuum random tree** (CRT) from a Brownian excursion. To produce tree, start with graph of Brownian excursion and then identify points connected by horizontal line segment that lies below graph except at endpoints. Result is a random matric space.
- Discrete analog: Consider a tree embedded in the plane with n edges and a distinguished root. As one traces the outer boundary of the tree clockwise, distance from root performs a simple walk on $\mathbf{Z}_+$ with $2n$ steps, starting and ending at 0.
- Simple bijection between rooted planar trees and walks of this type.
- CRT is in some sense the "uniformly random planar tree" of a given size.

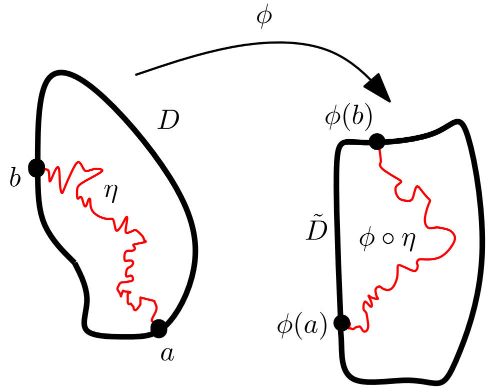
RANDOM PATHS

Given a simply connected planar domain D with boundary points a and b and a parameter $\kappa \in [0, \infty)$, the **Schramm-Loewner evolution** SLE_κ is a random non-self-crossing path in $\overline{D}$ from a to b.



The parameter κ roughly indicates how "windy" the path is. Would like to argue that SLE is in some sense the "canonical" random non-self-crossing path. What symmetries characterize SLE?

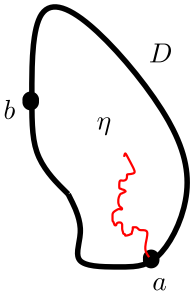
Conformal Markov property of SLE



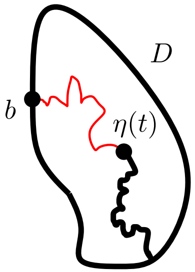
If ϕ conformally maps D to $\tilde{D}$ and η is an SLE_κ from a to b in D, then $\phi \circ \eta$ is an SLE_κ from $\phi(a)$ to $\phi(b)$ in $\tilde{D}$.

Markov Property

Given η up to a stopping time t...



law of remainder is SLE in $D \setminus \eta[0,t]$ from $\eta(t)$ to b.



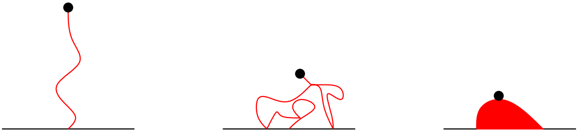
Chordal Schramm-Loewner evolution (SLE)

- **THEOREM [Oded Schramm]:** Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$.
- **Explicit construction:** An SLE path γ from 0 to ∞ in the complex upper half plane $\mathbf{H}$ can be defined in an interesting way: given path γ one can construct conformal maps $g_t : \mathbf{H} \setminus \gamma([0,t]) \to \mathbf{H}$ (normalized to look like identity near infinity, i.e., $\lim_{z \to \infty} g_t(z) - z = 0$). In SLE_κ, one defines g_t via an ODE (which makes sense for each fixed z):

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

where $W_t = \sqrt{\kappa} B_t =_{LAW} B_{\kappa t}$ and B_t is ordinary Brownian motion.

SLE phases [Rohde, Schramm]

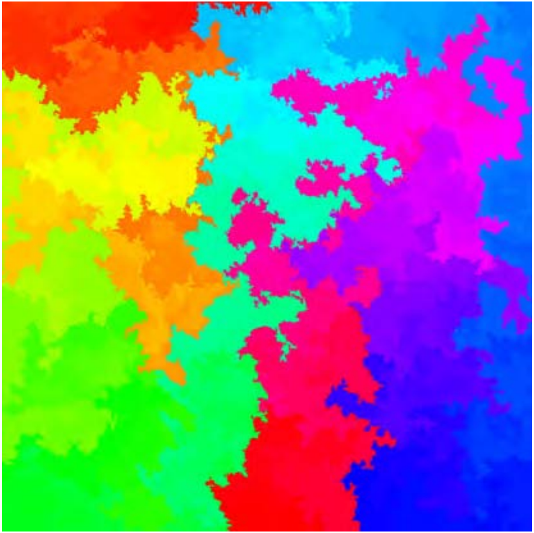


$\kappa \leq 4$ $\kappa \in (4,8)$ $\kappa \geq 8$

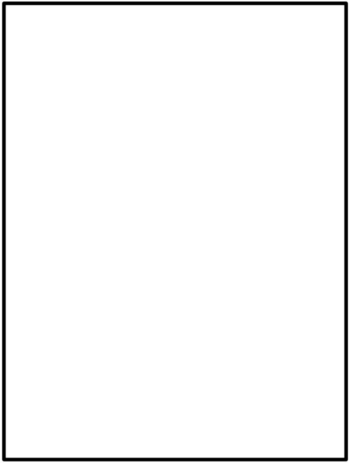
Radial Schramm-Loewner evolution (SLE)

- In *radial SLE* path grows from boundary of domain to center.
- Modified version allow growth from multiple boundary points (or a continuum of points) at once.
- This will be important when we think about continuum growth models.
- **Radial SLE:** $\partial_t g_t(z) = g_t(z)\frac{\xi_t+g_t(z)}{\xi_t-g_t(z)}$ where $\xi_t = e^{i\sqrt{\kappa}B_t}$.
- **Radial measure-driven Loewner evolution:** $\partial_t g_t(z) = \int g_t(z)\frac{x+g_t(z)}{x-g_t(z)}dm_t(x)$ where, for each g, m_t is a measure on the complex unit circle.

Continuum space-filling path

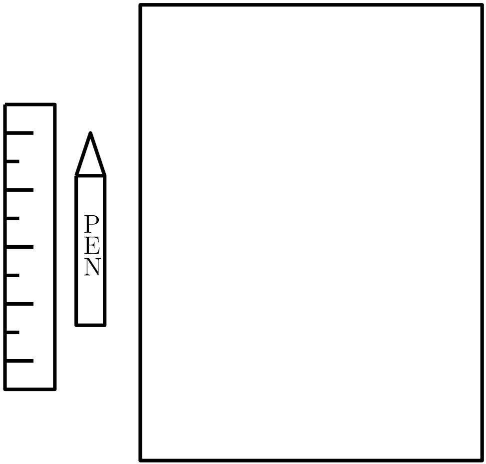


RANDOM SURFACES



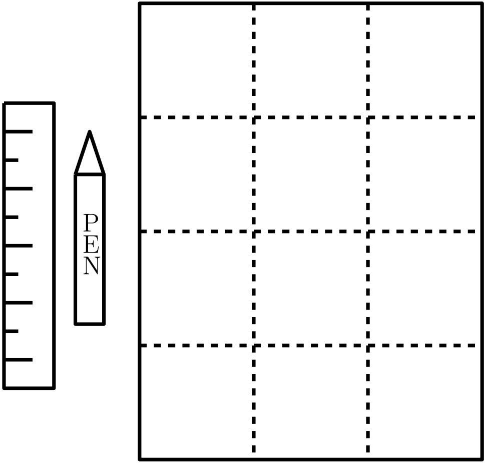
Start out with a sheet of paper

RANDOM SURFACES



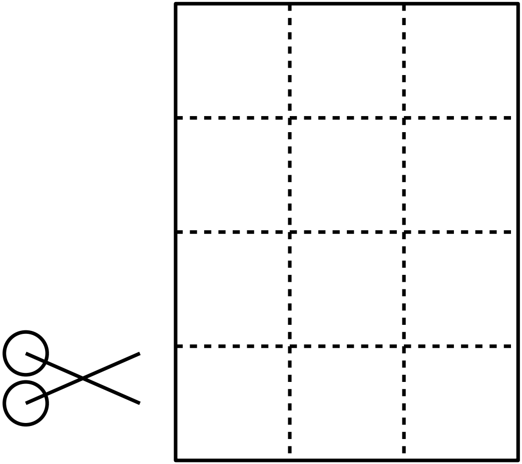
Get out pen and ruler

RANDOM SURFACES



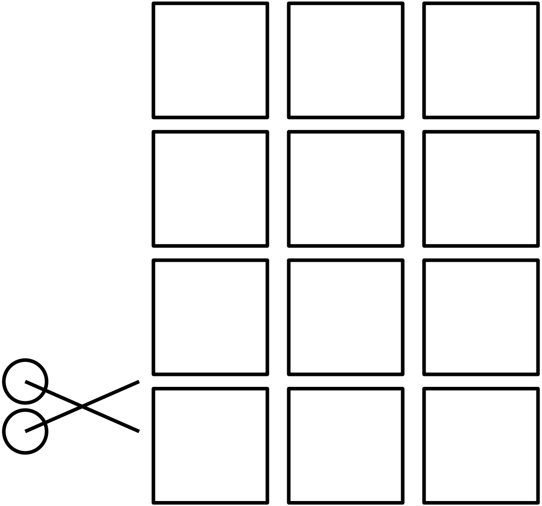
Measure and mark squares squares of equal size

RANDOM SURFACES



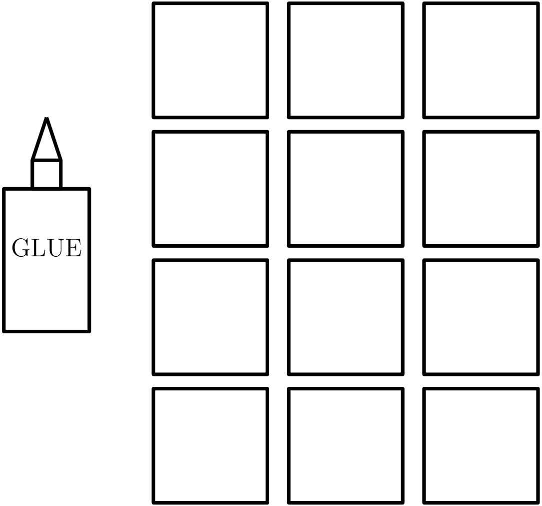
Get out scissors

RANDOM SURFACES



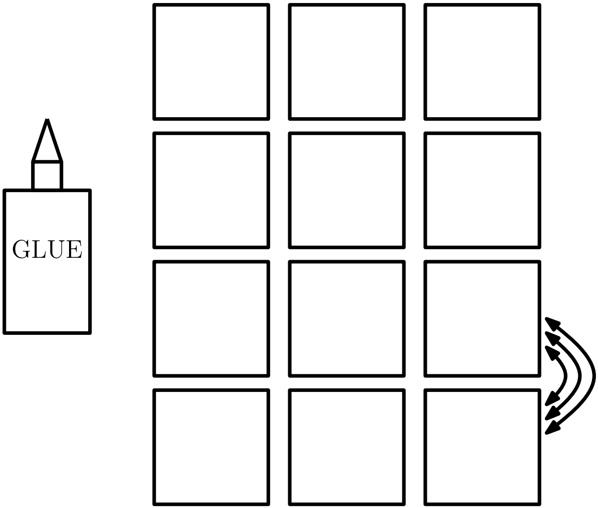
Cut into squares

RANDOM SURFACES



Get out bottle of glue

RANDOM SURFACES



Attach squares along boundaries with glue to form a surface "without holes."

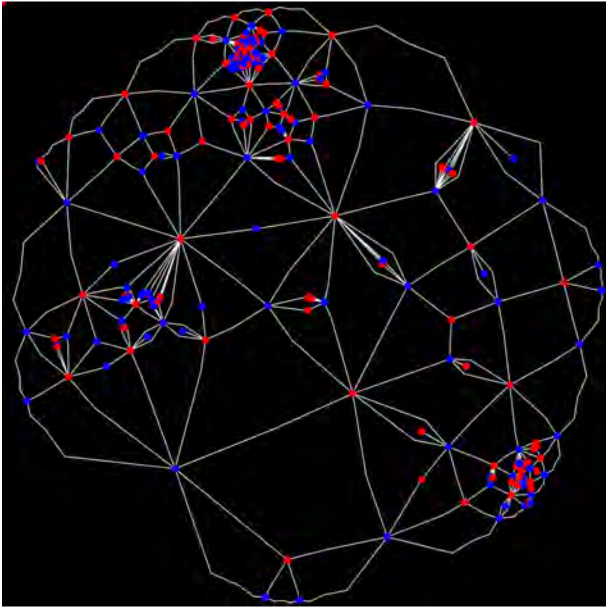
Random quadrangulation

Image removed due to copyright restrictions. See http://www.labri.fr/perso/marckert/simulations.html.

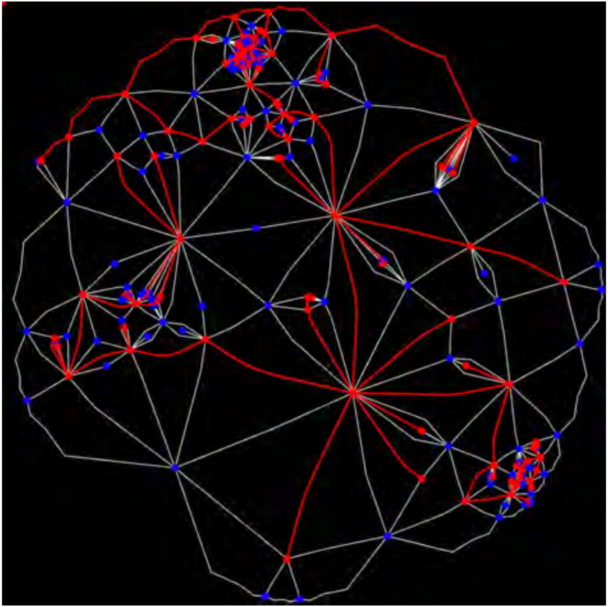
(Simulation due to J.F. Marckert)

1. First studied by Tutte in 1960s while working on the four color theorem.
2. Many variants (triangulations, quadrangulations, etc.) Some come equipped with extra statistical physics structure (a distinguished spanning tree, a general distinguished edge subset, a “spin” function on vertices, etc.)
3. Can be interpreted as Riemannian manifolds with conical singularities.
4. Converges in law in Gromov-Hausdorff sense to random metric space called Brownian map, homeomorphic to the 2-sphere, Hausdorff dimension 4 (established in several works by subsets of Chaissang, Schaefer, Le Gall, Paulin, Miermont)
5. Important tool: Bijections encoding surface via pair of trees.

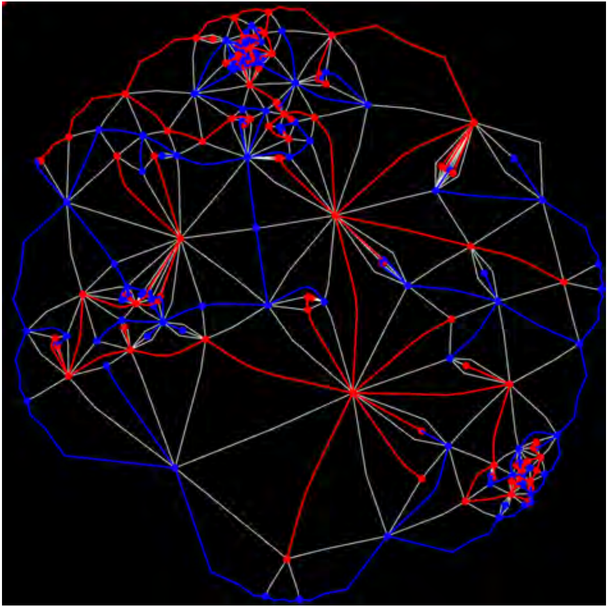
Random quadrangulation



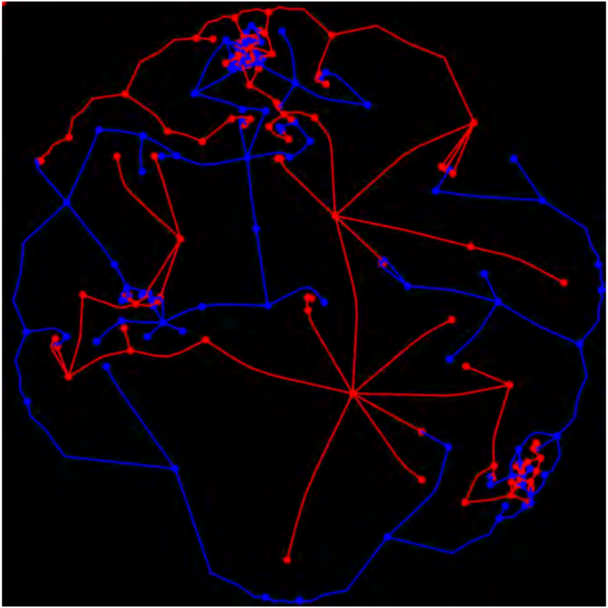
Red tree



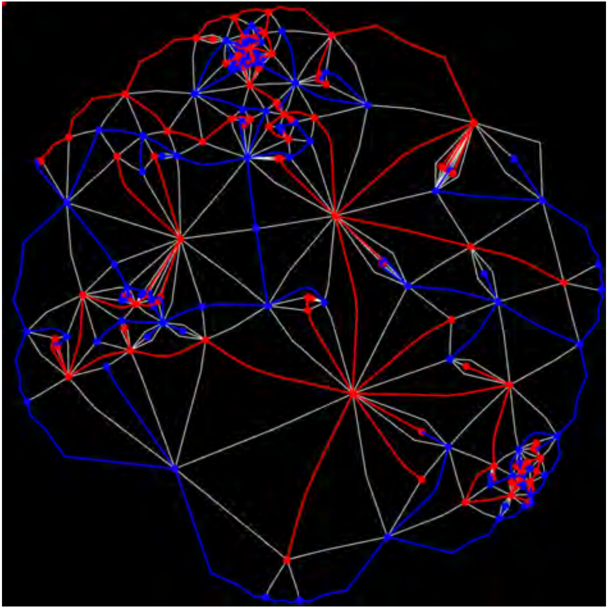
Red and blue trees



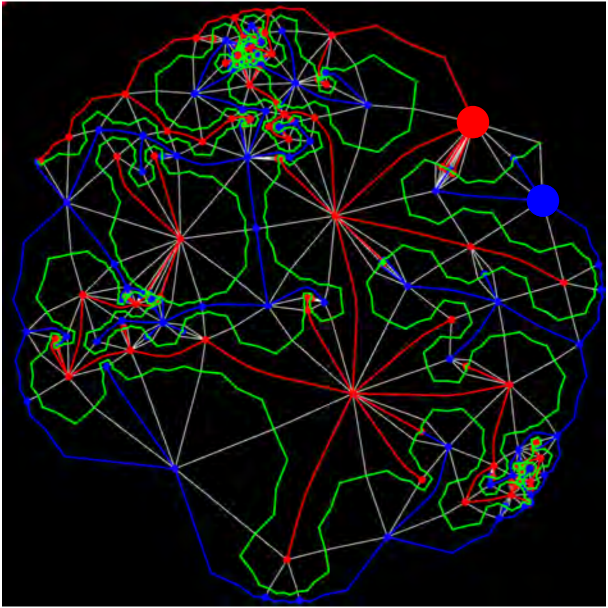
Red and blue trees alone do not determine the map structure



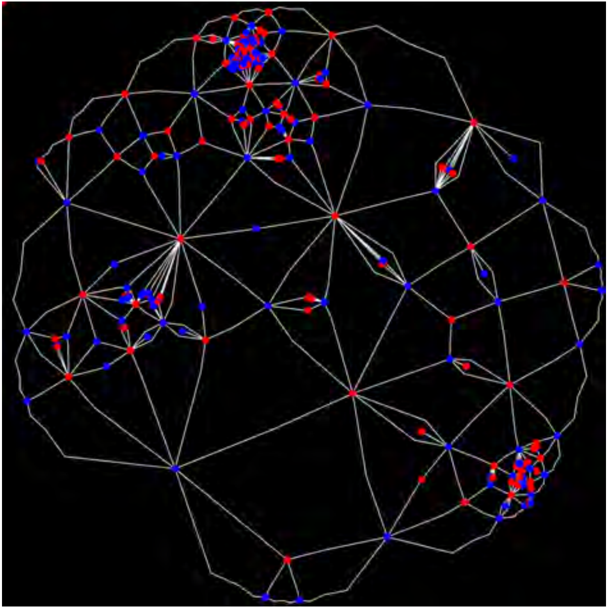
Random quadrangulation with red and blue trees



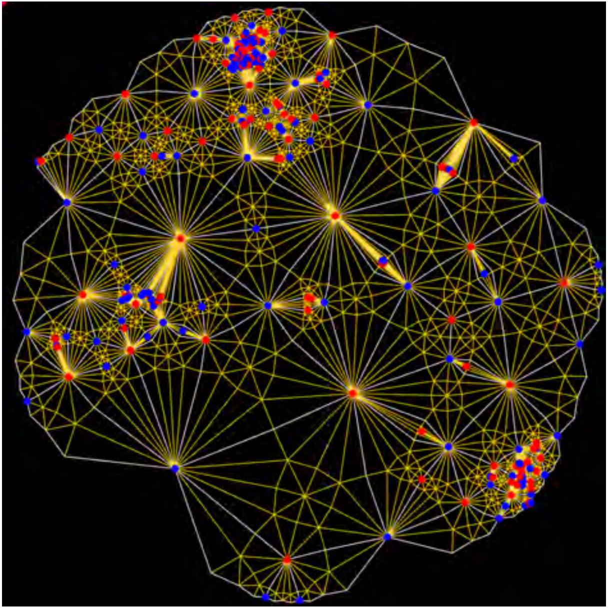
Path snaking between the trees. Encodes the trees and how they are glued together.



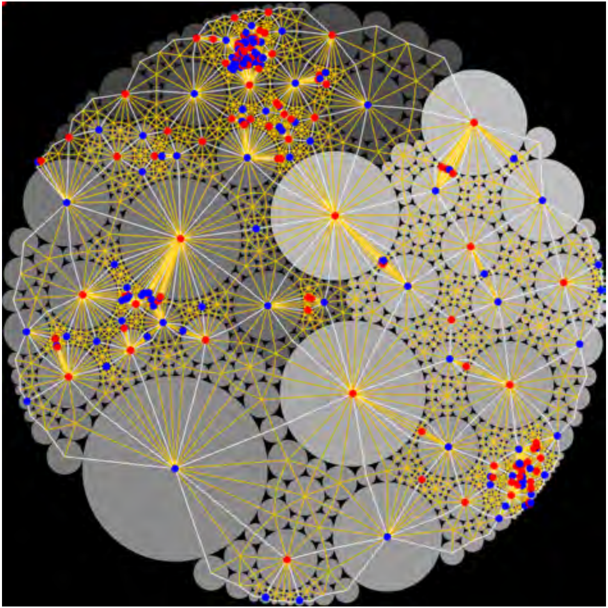
How was the graph embedded into $\mathbf{R}^2$?



Can subivide each quadrilateral to obtain a triangulation without multiple edges.

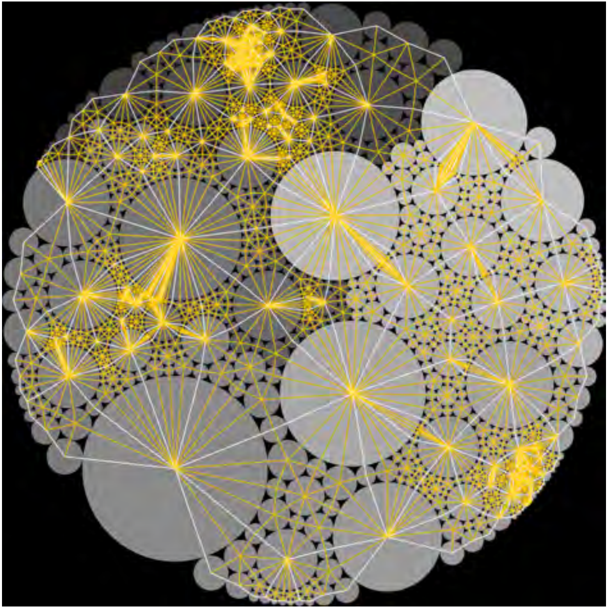


Circle pack the resulting triangulation.



Packed with Stephenson's CirclePack.

Circle pack the resulting triangulation.



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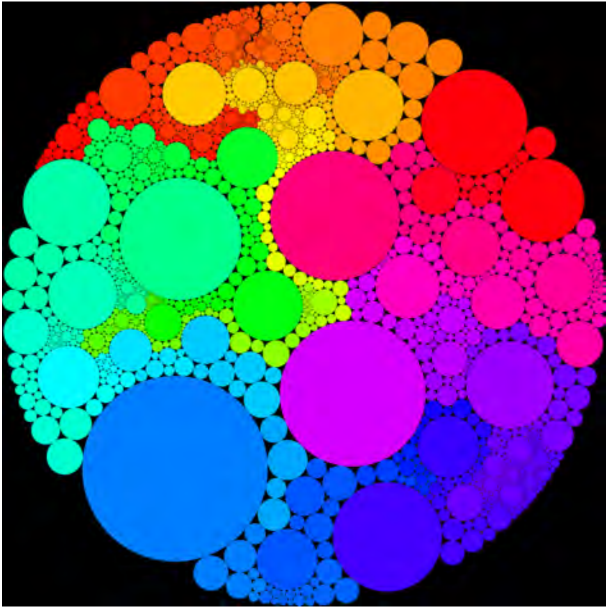
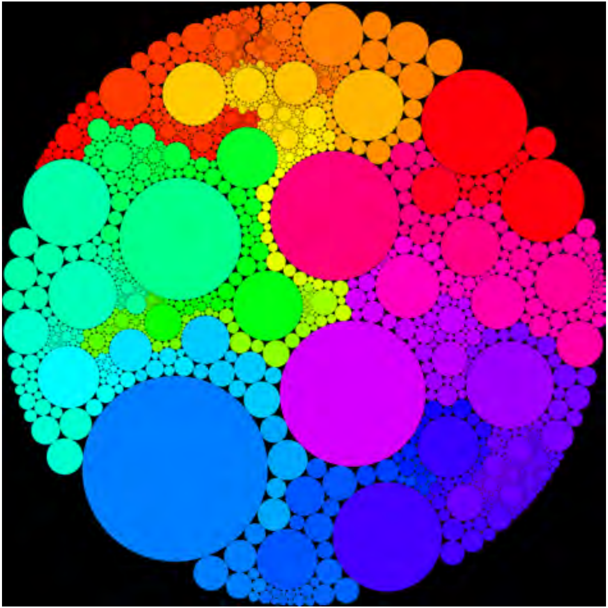


Packed with Stephenson's CirclePack.

What is the "limit" of this embedding? Circle packings are related to conformal maps.



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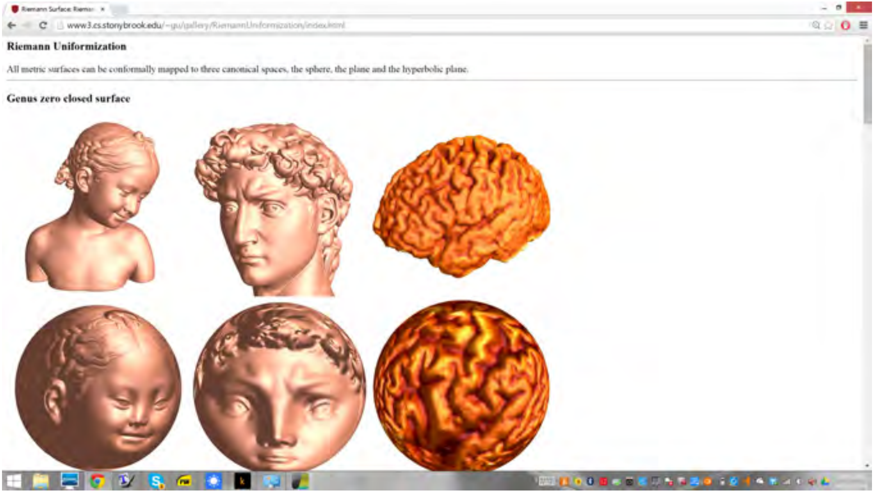
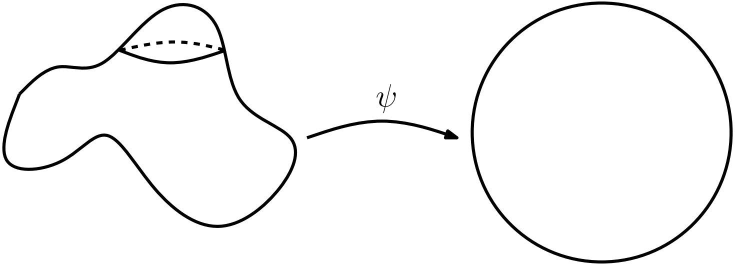
Conformal maps (from David Gu's web gallery)



Picking a surface at random in the continuum

Uniformization theorem: every simply connected Riemannian surface can be conformally mapped to either the unit disk, the plane, or the sphere $\mathbf{S}^2$ in $\mathbf{R}^3$



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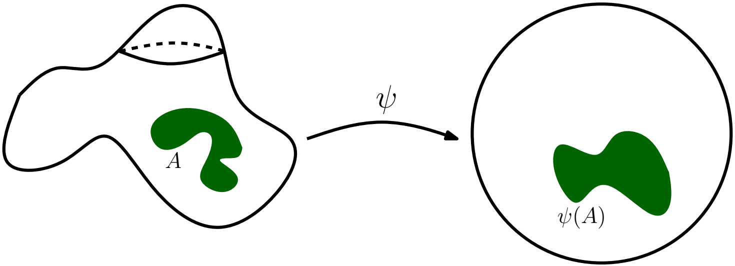


Isothermal coordinates: Metric for the surface takes the form $e^{\rho(z)}dz$ for some smooth function ρ where dz is the Euclidean metric.

$\Rightarrow$ Can parameterize the space of surfaces with smooth functions.

- If $\rho = 0$, get the same surface
- If $\Delta\rho = 0$, i.e. if ρ is harmonic, the surface described is flat

Question: Which measure on ρ? If we want our surface to be a perturbation of a flat metric, natural to choose ρ as the canonical perturbation of a harmonic function.

The Gaussian free field

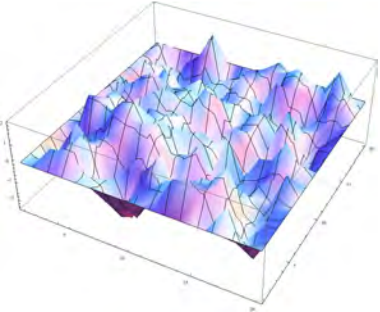
- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.
- Measure on functions $h\colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ with density respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

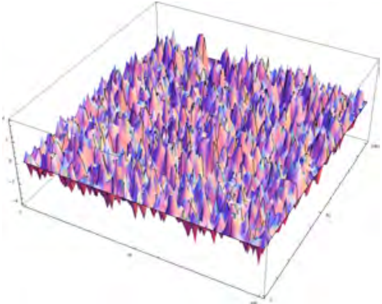
$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2}\sum_{x\sim y}(h(x)-h(y))^2\right)$$

- Natural perturbation of a harmonic function
- Fine mesh limit: converges to the continuum GFF, i.e. the standard Gaussian wrt the **Dirichlet inner product**

$$(f,g)_\nabla = \frac{1}{2\pi}\int \nabla f(x)\cdot\nabla g(x)dx.$$

- Continuum GFF not a function — only a generalized function

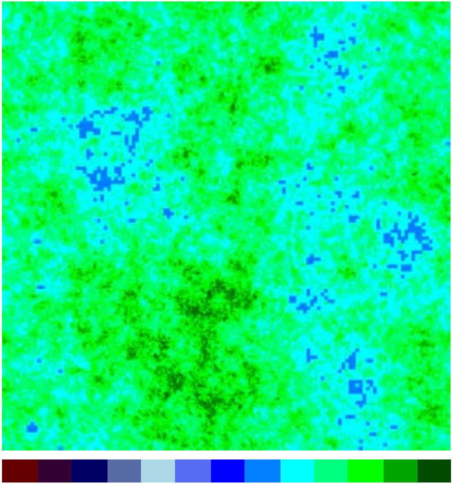




Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)}dz$ where h is a GFF and $\gamma \in [0,2)$
- Random surface model: Polyakov, 1980. Motivated by string theory.
- Rigorous construction of measure: Høegh-Krohn, 1971, $\gamma \in [0,\sqrt{2})$. Kahane, 1985, $\gamma \in [0,2)$.
- Does not make literal sense since h takes values in the space of distributions.
- Can make sense of random area measure using a regularization procedure.

$\gamma = 0.5$



(Number of subdivisions)

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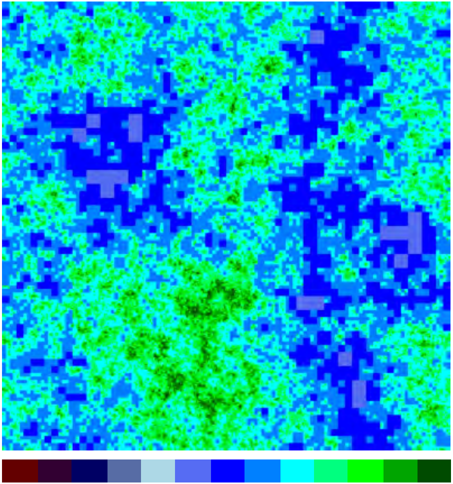
$\gamma = 1.0$



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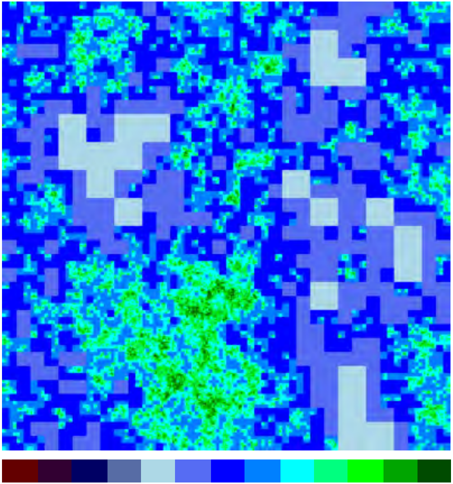

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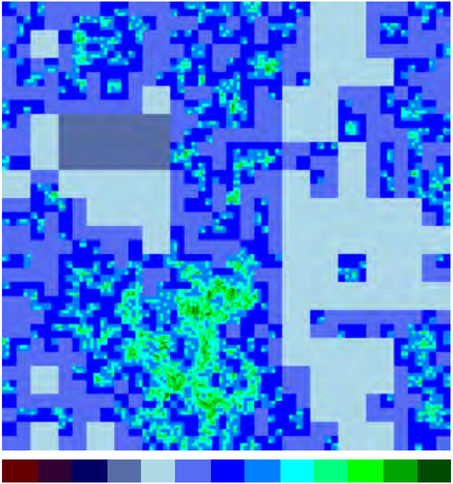
$\gamma = 2.0$



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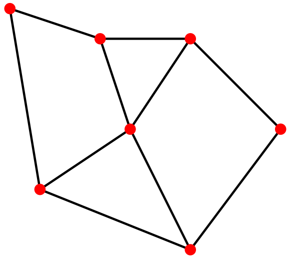
RANDOM GROWTH

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)



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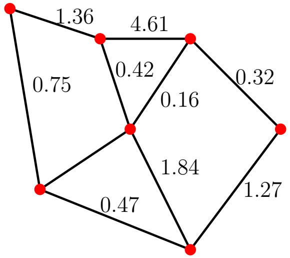
- Associate with a graph (V, E) i.i.d. $\exp(1)$ edge weights



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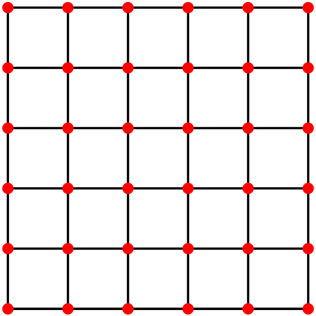

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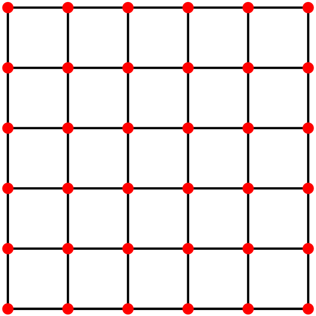

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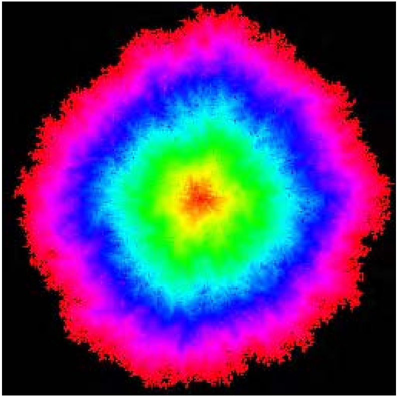
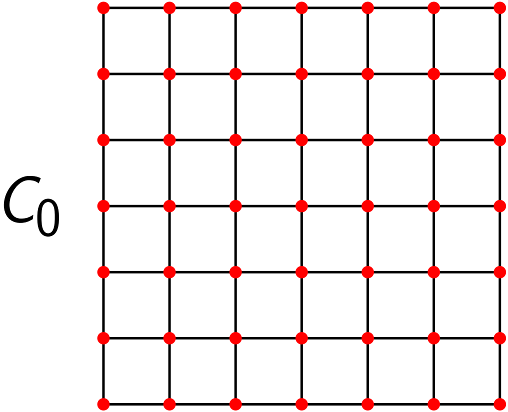
- **Question:** Large scale behavior of shape of ball wrt perturbed metric?
- Cox and Durrett (1981) showed that the macroscopic shape is convex
- Computer simulations show that it is not a Euclidean disk
- $\mathbf{Z}^2$ is not isotropic enough
- Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if $\mathbf{Z}^2$ is replaced by the Voronoi tesselation associated with a Poisson process



Markovian formulation

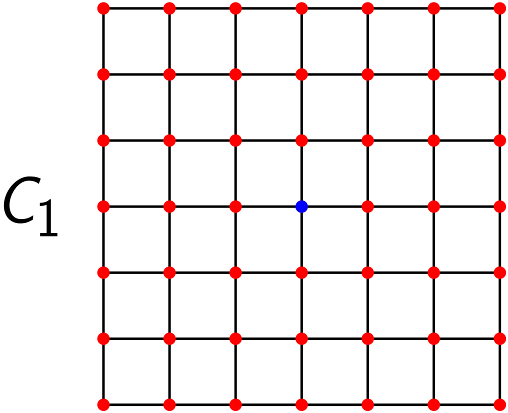
Eden exploration



Sample the cluster C_{n+1} from C_n by selecting an edge uniformly at random on ∂C_n, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to η power (η-DBM).

Markovian formulation

Eden exploration



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Markovian formulation

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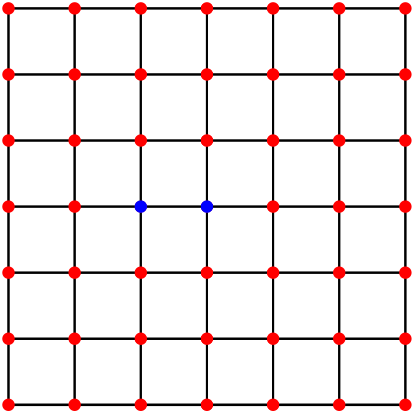
C_2



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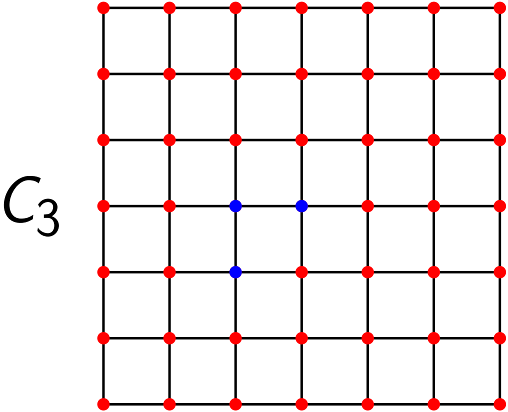

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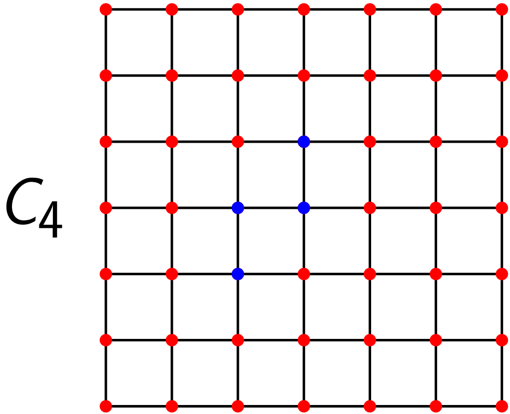

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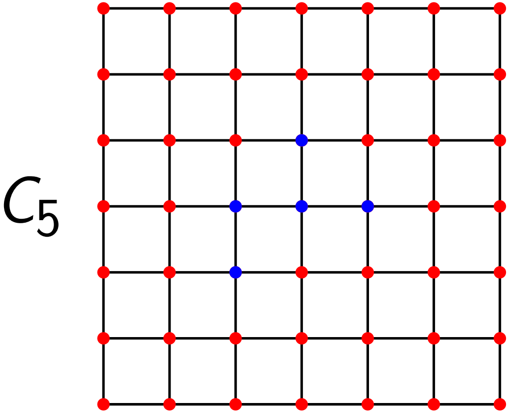

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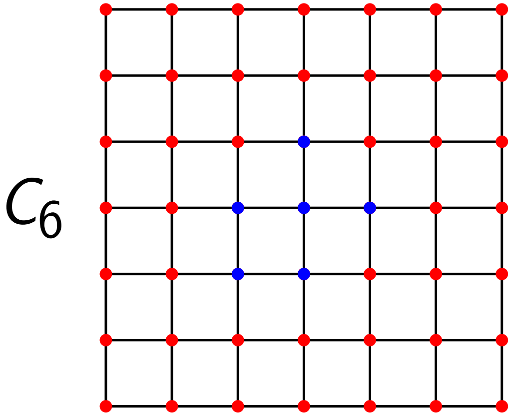

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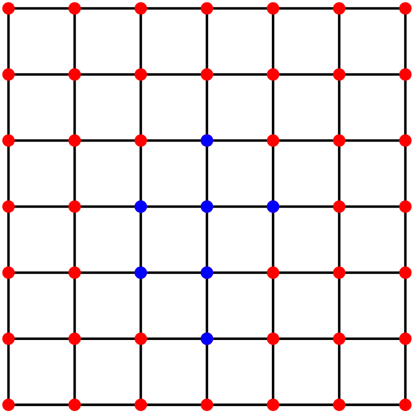
Markovian formulation

Eden exploration

C_7



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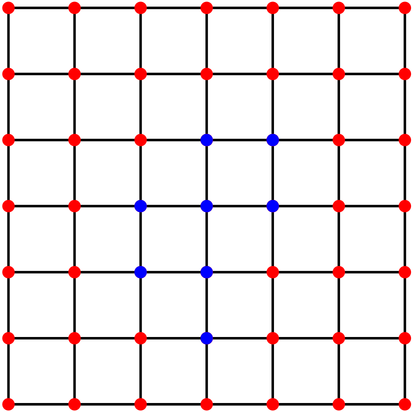
C_8



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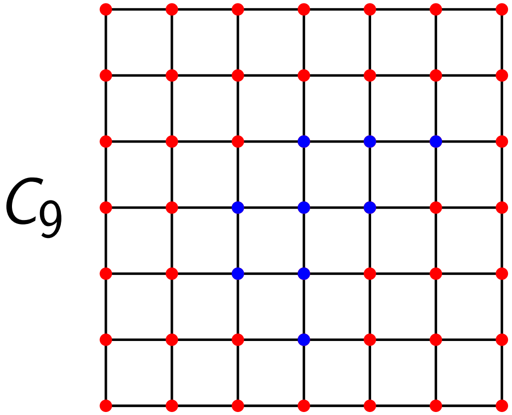

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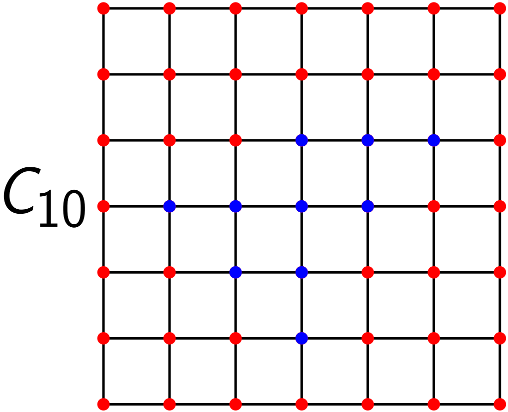

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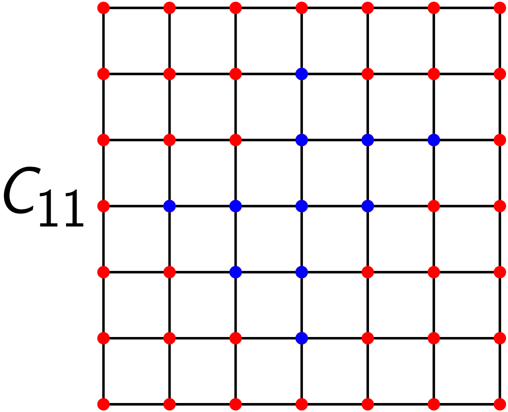

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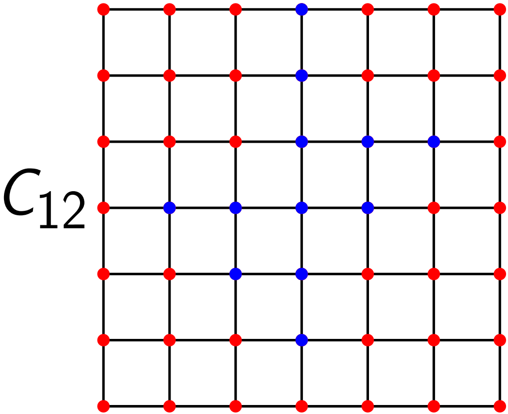

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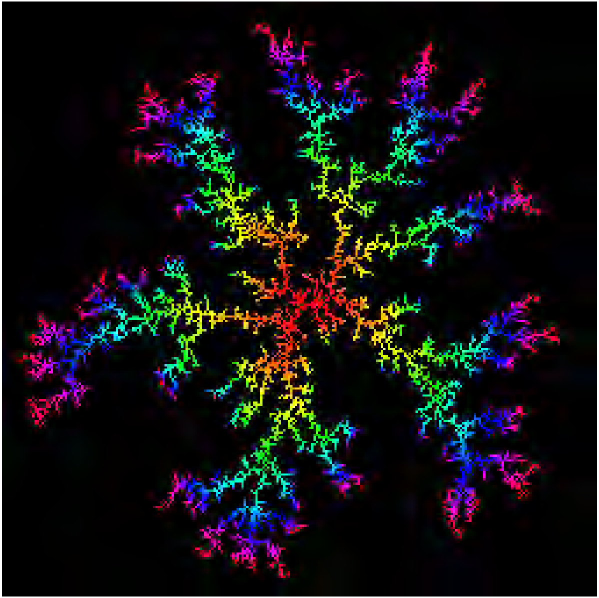


Euclidean Diffusion Limited Aggregation (DLA) introduced by Witten-Sander 1981.

DLA in nature: "A DLA cluster grown from a copper sulfate solution in an electrodeposition cell" (from Wikipedia)

DLA in nature: Magnese oxide patterns on the surface of a rock. (Halsey, Physics Today 2000)

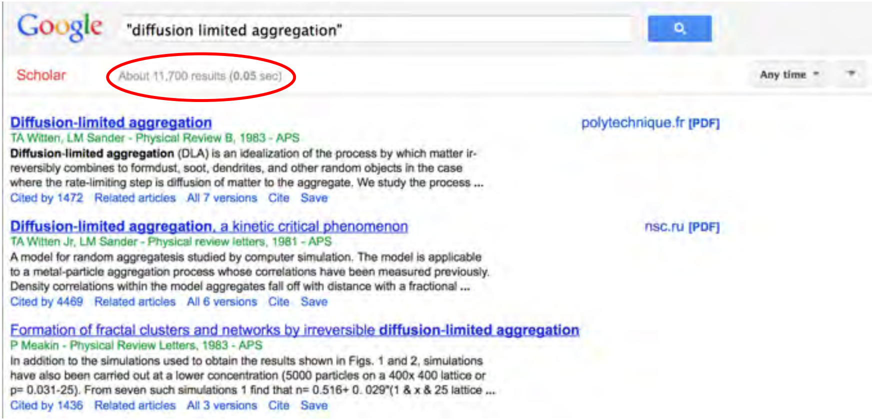
DLA in nature: Magnese oxide patterns on the surface of a rock.

DLA in art: "High-voltage dielectric breakdown within a block of plexiglas" (from Wikipedia)

DLA in physics

Introduced by Witten and Sander in 1981 as a model for crystal growth. (Mineral deposits, Hele-Shaw flow, electrodeposition, lichen growth, lightning paths, coral, etc.)

An active area of research in physics for the last 33 years:



DLA in math?

Not a lot of progress. (A related process called internal DLA is mathematically much more well understood.) Expected that (as with Eden model) lattice versions may have anisotropic features in limit.

Open questions

- Does DLA have a "scaling limit"?
- Is the shape random at large scales?
- Does the macroscopic shape look like a tree?
- What is its asymptotic dimension? Simulation prediction: ≈ 1.71 on $\mathbf{Z}^2$
- Is there a *universal* isotropic continuum analog of DLA?

What about DLA on random planar maps and Liouville quantum gravity surfaces?

Part II:
DRAMA

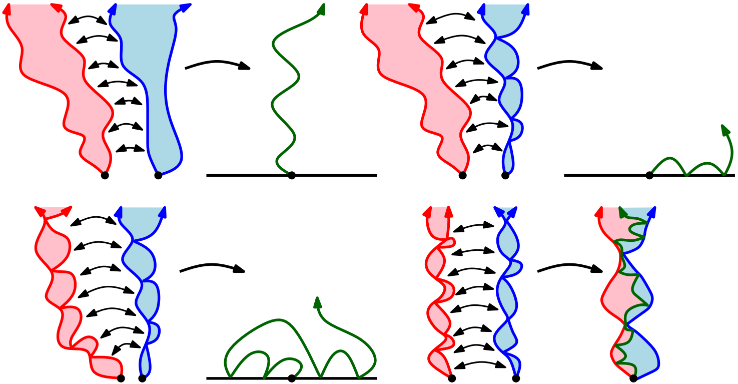
STORY A:

SURFACE PLUS SURFACE =
SURFACE PLUS CURVE

independence on both sides

WELDING RANDOM SURFACES

Can "weld" and "slice" special quantum surfaces called quantum wedges (with "weight" parameters indicating thickness) to obtain wedges (with other weights).



- Weight parameter $W = \gamma(\gamma + \frac{2}{\gamma} - \alpha)$ is additive under the welding operation.
- Interface between welding of independent wedges $\mathcal{W}_1, \mathcal{W}_2$ of weight W_1 and W_2 is an $\mathrm{SLE}_\kappa(W_1 - 2; W_2 - 2)$ on combined surface.
- Glue **canonical random surfaces**, seam becomes **canonical random path**.

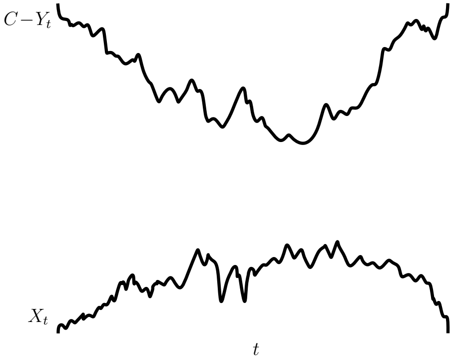
STORY B:

TREE PLUS TREE =
SURFACE PLUS
SPACE-FILLING CURVE

LHS independent or correlated,
RHS independent

MATING RANDOM TREES

X, Y independent Brownian excursions on $[0,1]$. Pick $C > 0$ large so that the graphs of X and $C - Y$ are disjoint.



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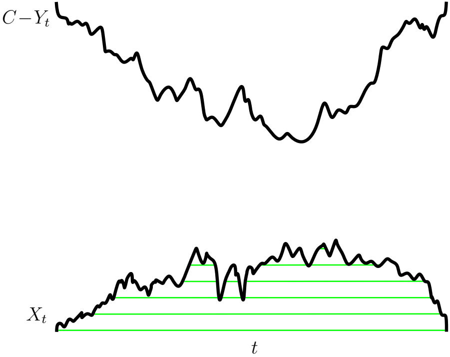

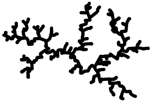

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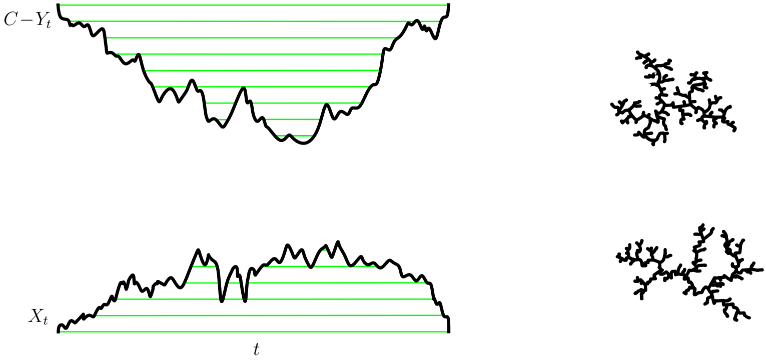

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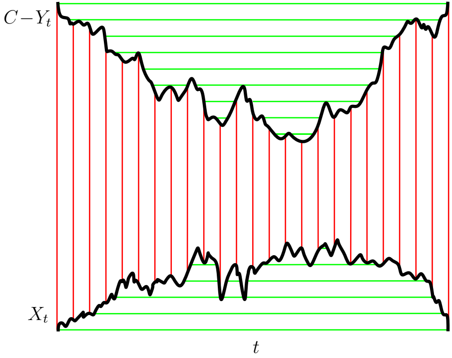

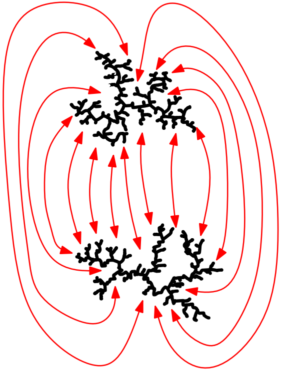

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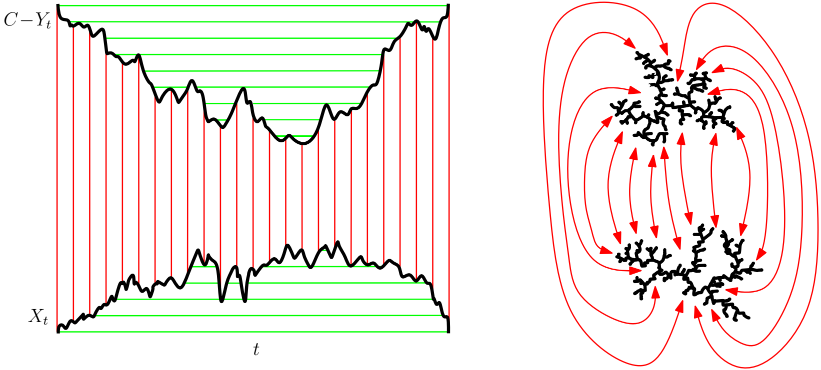

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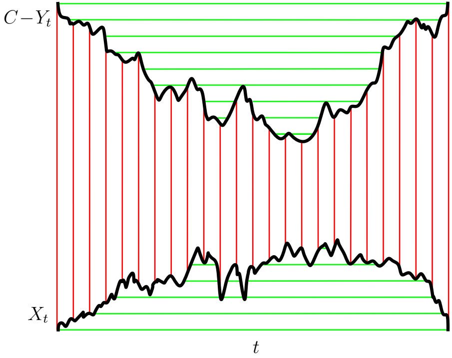

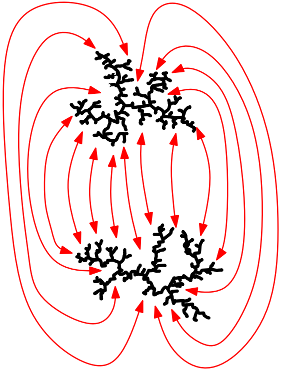

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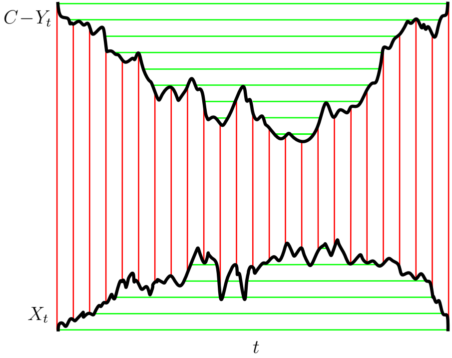

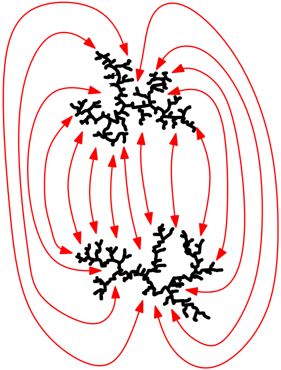

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Q: What is the resulting structure? **A:** Sphere with a space-filling path. A **peanosphere**.

Surface is topologically a sphere by Moore's theorem

Theorem (Moore 1925)

Let $\cong$ be any topologically closed equivalence relation on the sphere $\mathbf{S}^2$. Assume that each equivalence class is connected and not equal to all of $\mathbf{S}^2$. Then the quotient space $\mathbf{S}^2/\cong$ is homeomorphic to $\mathbf{S}^2$ if and only if no equivalence class separates the sphere into two or more connected components.

- An equivalence relation is topologically closed iff for any two sequences (x_n) and (y_n) with
  - $x_n \cong y_n$ for all n
  - $x_n \to x$ and $y_n \to y$
- we have that $x \cong y$.

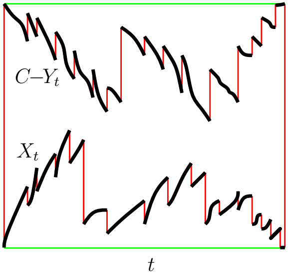
STORY C:

SURFACE TREE PLUS SURFACE TREE $=$ SURFACE PLUS SELF-HITTING CURVE

independence on both sides

Gluing independent Lévy trees

Can view $\mathrm{SLE}_{\kappa'}$ process, $\kappa' \in (4, 8)$ as a gluing of two $\frac{\kappa'}{4}$-stable Lévy trees.



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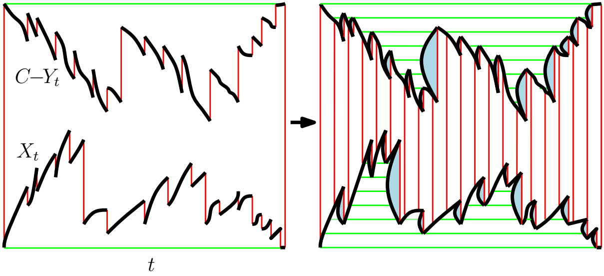

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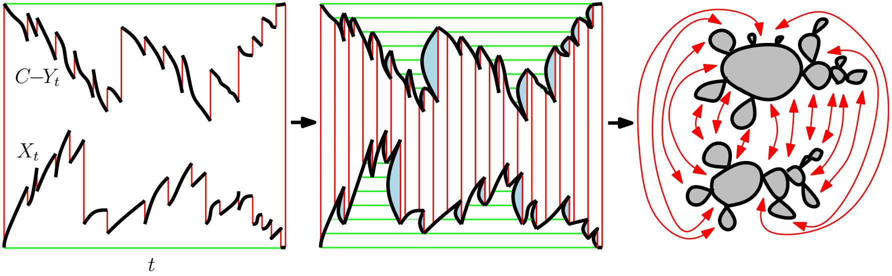

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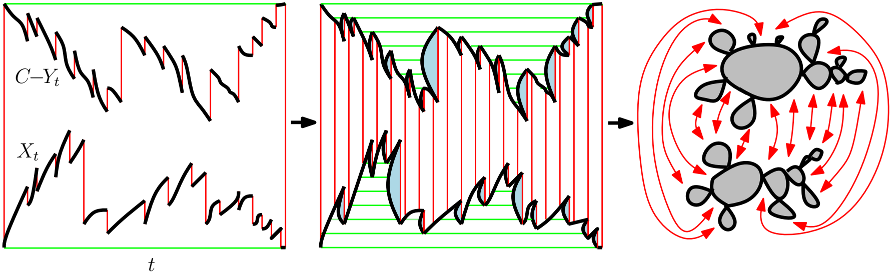


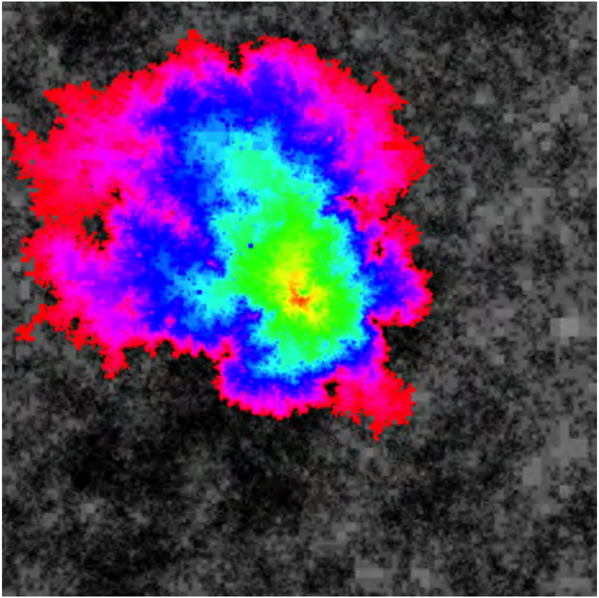
- The two trees of quantum disks almost surely determine both the $\mathrm{SLE}_{\kappa'}$ and the LQG surface on which it is drawn
- Can convert questions about $\mathrm{SLE}_{\kappa'}$ into questions about $\frac{\kappa'}{4}$-stable processes.
- Scaling limit of "exploration path" on random planar map should be SLE_6 on a $\sqrt{8/3}$-LQG. Using welding machinery, we can understand well the "bubbles" cut out by such an exploration process. We can understand conditional law of unexplored region given what we have seen.

STORY D:

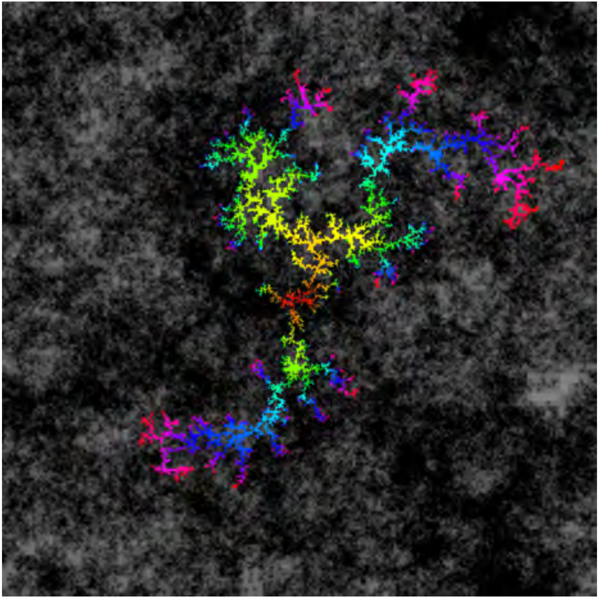
GROWTH ON SURFACE $=$
"RESHUFFLED" CURVE
ON SURFACE

RANDOM GROWTH ON RANDOM SURFACES

- Can we make sense of η-DBM on a γ-LQG? We have shown how to tile an LQG surface with diadic squares of "about the same size" so we could run a DLA on this set of squares and try to take a fine mesh limit.
- Or we could try η-DBM on corresponding RPM, which one would expect to behave similarly....
- **Question:** Are there coral reefs, snowflakes, lichen, crystals, plants, lightning bolts, etc. whose growth rates are affected by a random medium (something like LQG)? The simulations look similar but have a bit more personality when γ is larger (as we will see). They look like Chinese dragons.
- We will ultimately want to construct a candidate for the scaling limit, which we will call (for reasons explained later) **quantum Loewner evolution:** QLE(γ^2, η).
- But first let's look at some computer generated images (and some animations), starting with an Eden exploration.



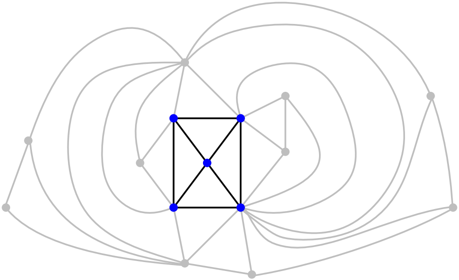
Eden model on $\sqrt{8/3}$-LQG



DLA on a $\sqrt{2}$-LQG

Eden model on planar map

- Random planar map, random vertex x. Perform FPP from x.



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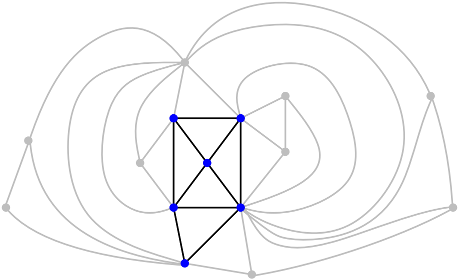

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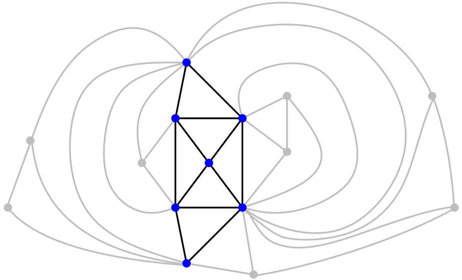

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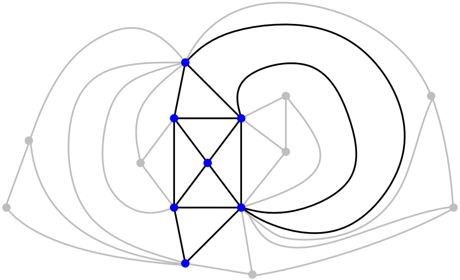

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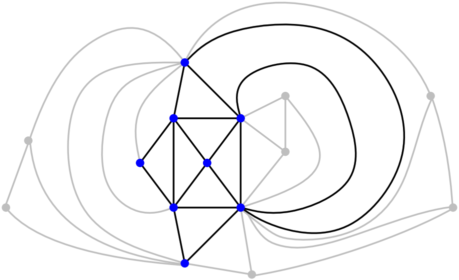

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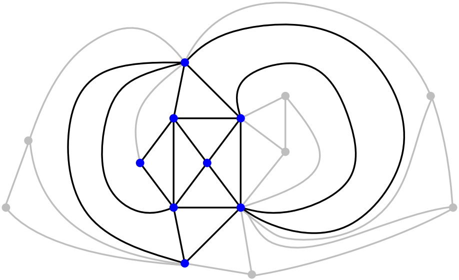

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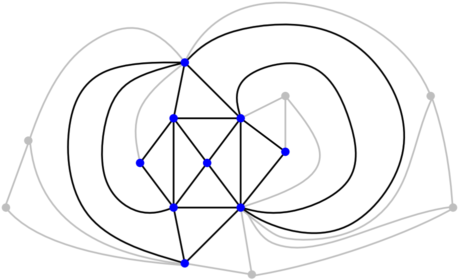

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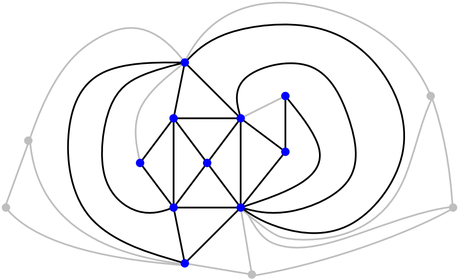

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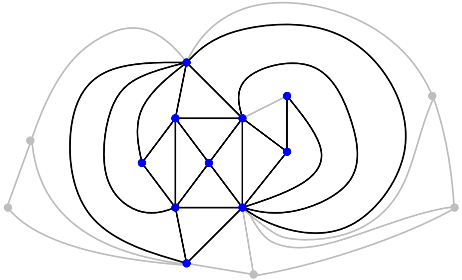

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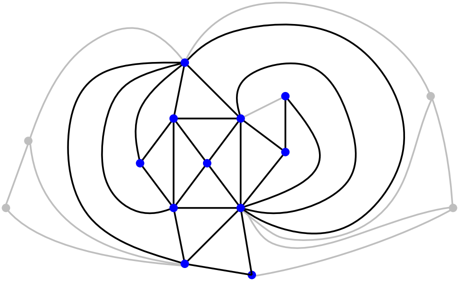

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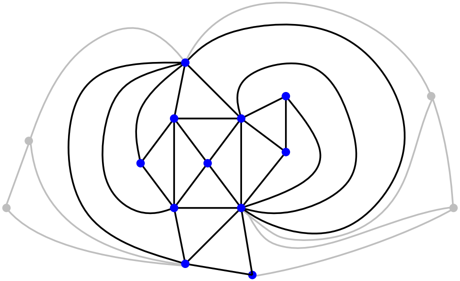

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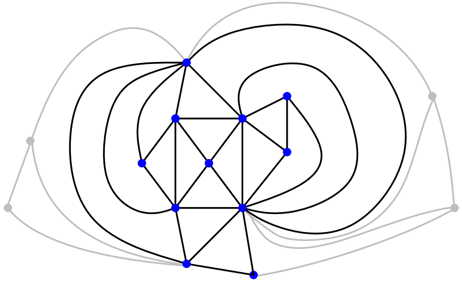


Important observations:

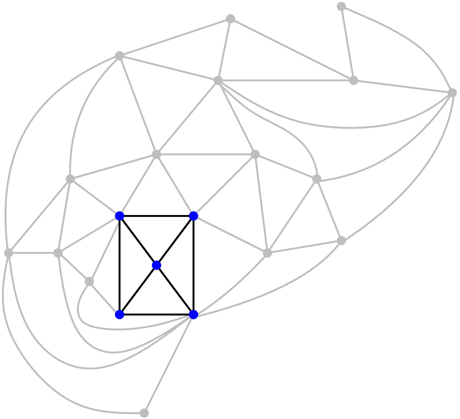
- Conditional law of map given ball at time n only depends on the boundary lengths of the outside components. *Exploration respects the Markovian structure of the map.*
- If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length

Belief: Isotropic enough so that at large scales this is close to a ball in the graph metric

First passage percolation on random planar maps III

Variant:

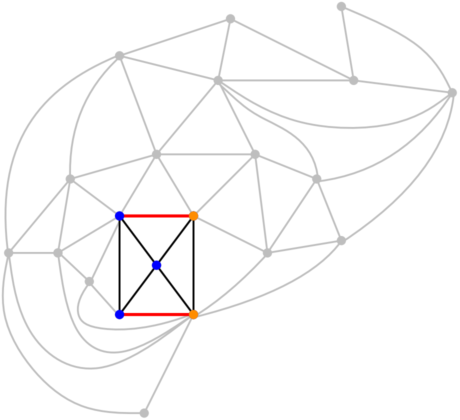
- Pick two edges on outer boundary of cluster



First passage percolation on random planar maps III

Variant:

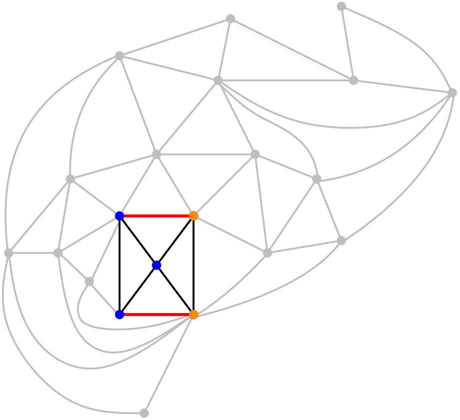
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow



First passage percolation on random planar maps III

Variant:

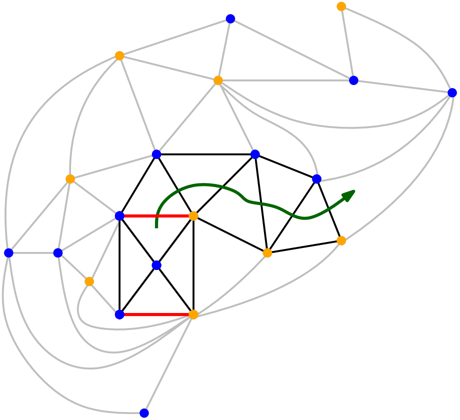
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$



First passage percolation on random planar maps III

Variant:

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface



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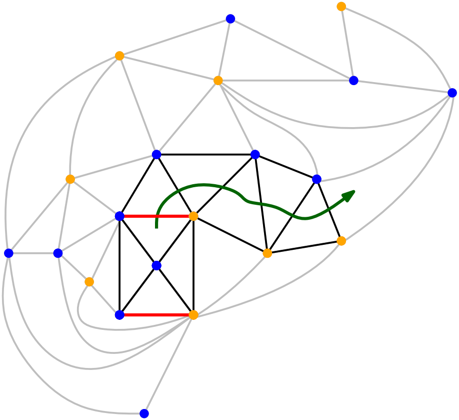

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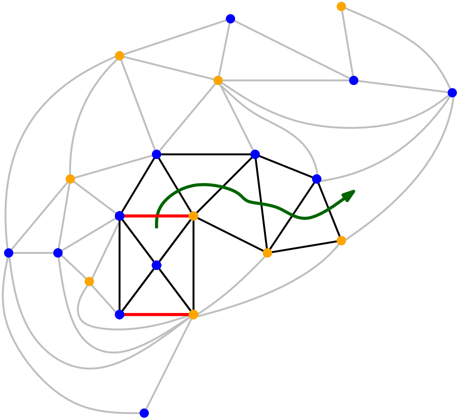

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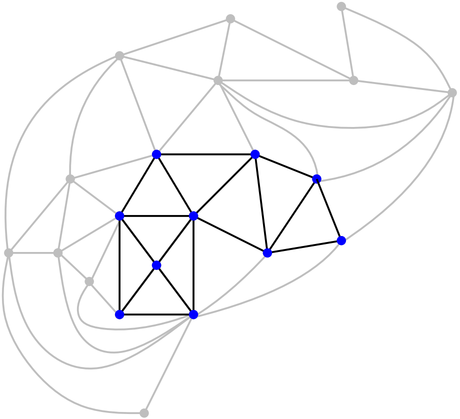

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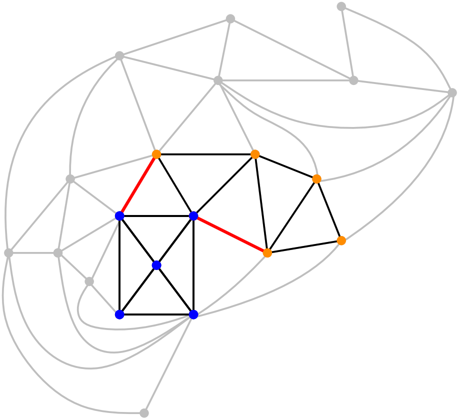

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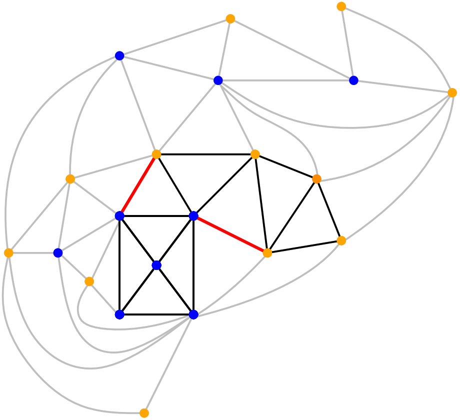

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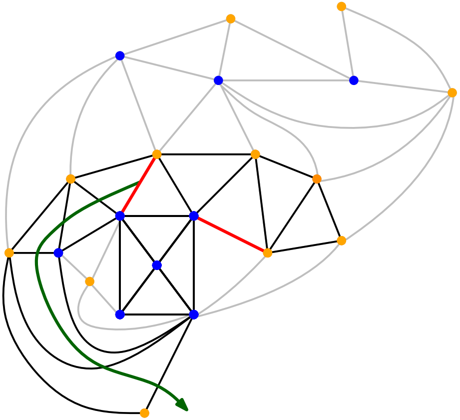

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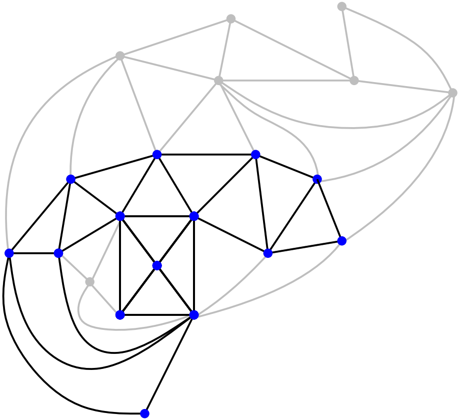

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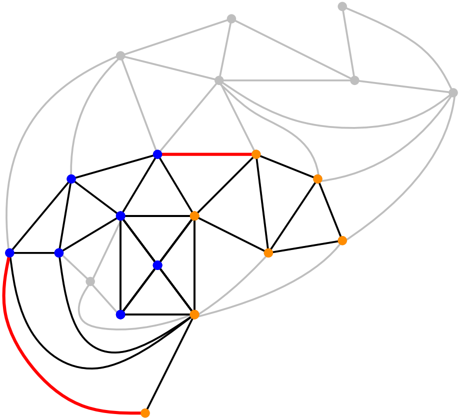

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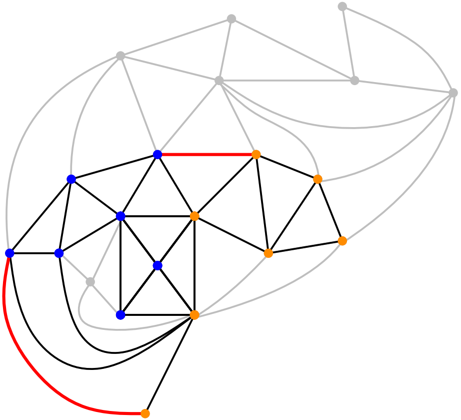
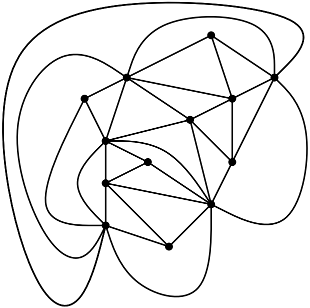


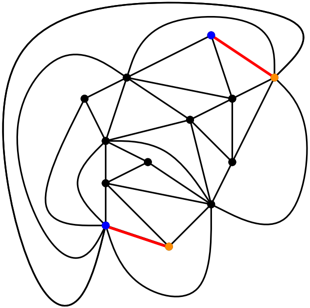
- *This exploration also respects the Markovian structure of the map.*
- If we work on an “infinite” planar map, the conditional law of the map in the unbounded component only depends on the boundary length.
- Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball

Continuum limit ansatz



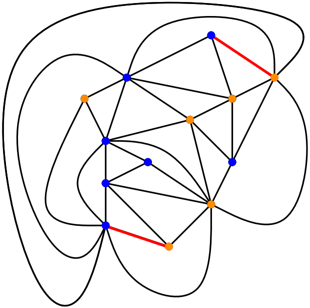
- Sample a random planar map

Continuum limit ansatz



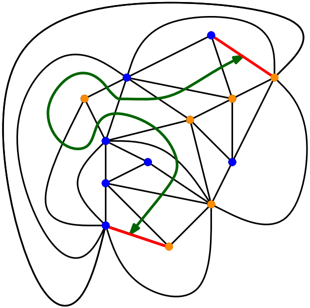
- Sample a random planar map and two edges uniformly at random

Continuum limit ansatz



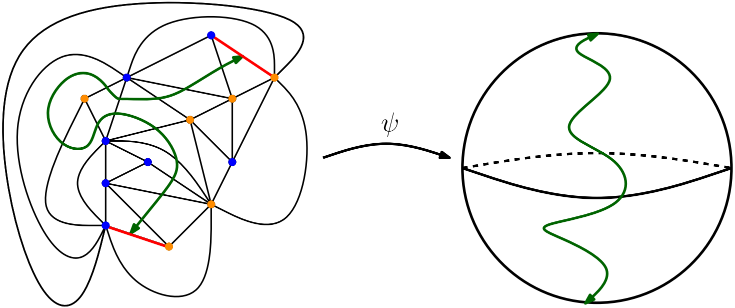
- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability 1/2

Continuum limit ansatz



- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability 1/2 and draw percolation interface

Continuum limit ansatz



- Sample a random planar map and two edges uniformly at random
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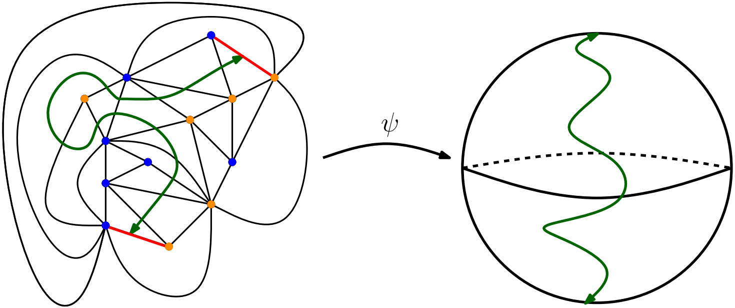
- Conformally map to the sphere

Continuum limit ansatz



- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability $1/2$ and draw percolation interface
- Conformally map to the sphere

Ansatz Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent SLE_6.

Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x

Continuum analog of first passage percolation on LQG

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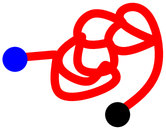
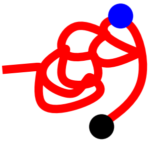


Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6
- Resample the tip according to boundary length



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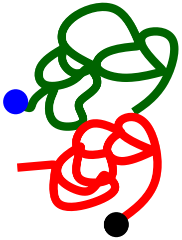
- Repeat



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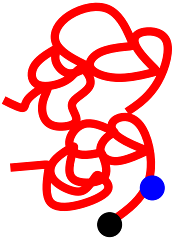

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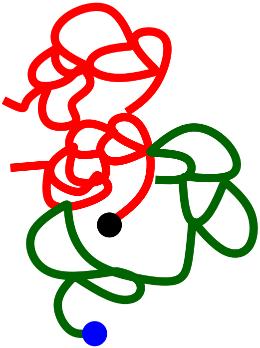

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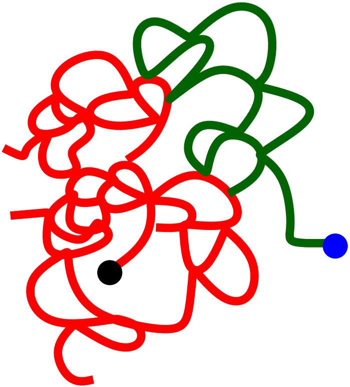

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Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
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- Repeat
- Know the conditional law of the LQG surface at each stage, using exploration results

Continuum analog of first passage percolation on LQG

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$\mathrm{QLE}(8/3, 0)$ is the limit as $\delta \to 0$ of this growth process. It is described in terms of a radial Loewner evolution which is driven by a measure valued diffusion.

Continuum analog of first passage percolation on LQG

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$\mathrm{QLE}(8/3, 0)$ is the limit as $\delta \to 0$ of this growth process. It is described in terms of a radial Loewner evolution which is driven by a measure valued diffusion.

$\mathrm{QLE}(8/3, 0)$ is SLE_6 with **tip re-randomization**. It can be understood as a "reshuffling" of the exploration procedure associated to the peanosphere.

What is $\mathrm{QLE}(\gamma^2, \eta)$?

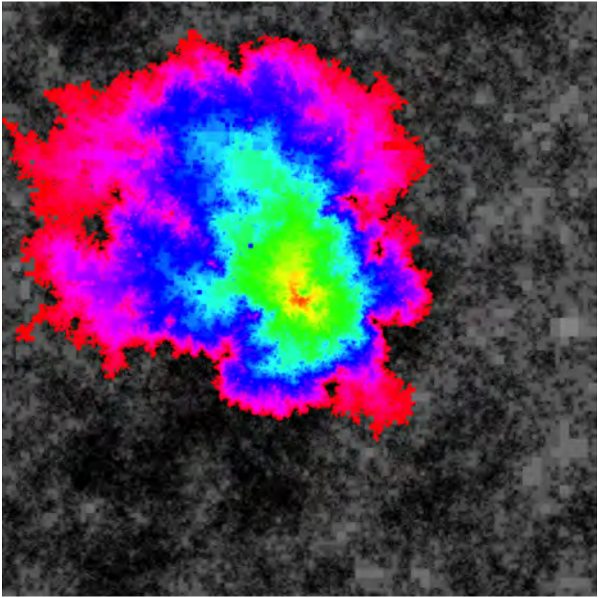
$\mathrm{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\mathrm{QLE}(\gamma^2, \eta)$

- γ is the type of LQG surface on which the process grows
- η determines the manner in which it grows

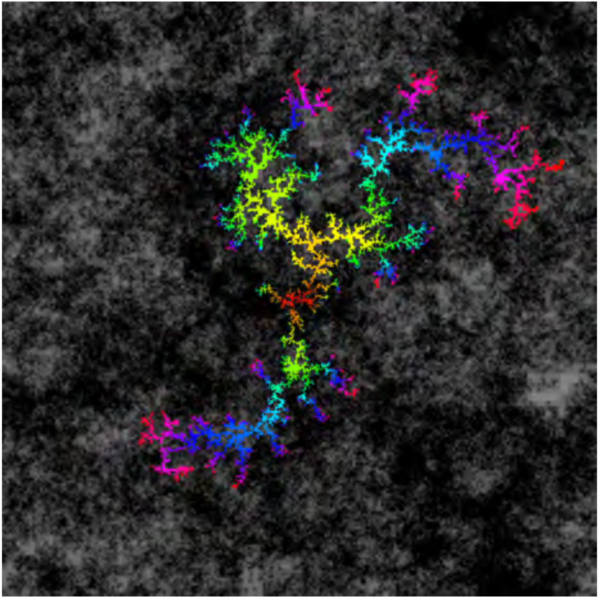
Let μ_{HARM} (resp. μ_{LEN}) be harmonic (resp. length) measure on a γ-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

$$\left(\frac{d\mu_{\mathrm{HARM}}}{d\mu_{\mathrm{LEN}}}\right)^{\eta} d\mu_{\mathrm{LEN}}.$$

- **First passage percolation:** $\eta = 0$
- **Diffusion limited aggregation:** $\eta = 1$
- **η-dieletric breakdown model:** general values of η

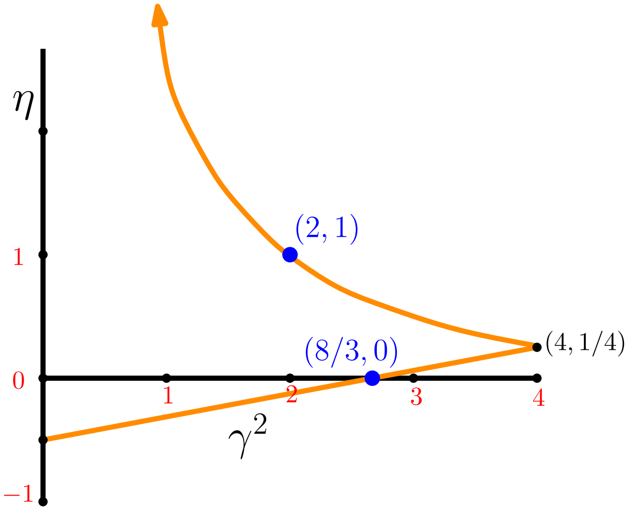


Discrete approximation of $\mathrm{QLE}(8/3, 0)$. Metric ball on a $\sqrt{8/3}$-LQG



Discrete approximation of $\mathrm{QLE}(2,1)$. DLA on a $\sqrt{2}$-LQG

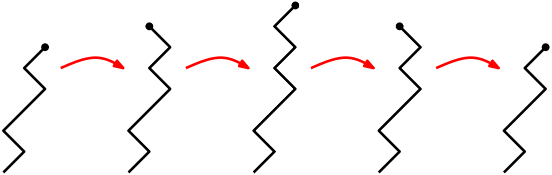
$\mathrm{QLE}(\gamma^2, \eta)$ processes we can construct



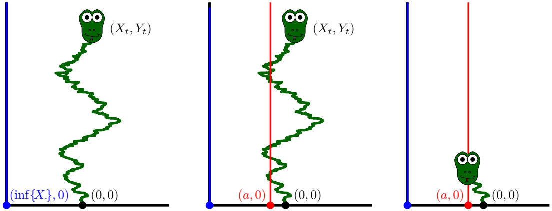
Each of the $\mathrm{QLE}(\gamma^2, \eta)$ processes with (γ^2, η) on the orange curves is built from an SLE_κ process using tip re-randomization.

STORY E:

BROWNIAN MAP $=$ $\sqrt{8/3}$-LIOUVILLE QUANTUM GRAVITY



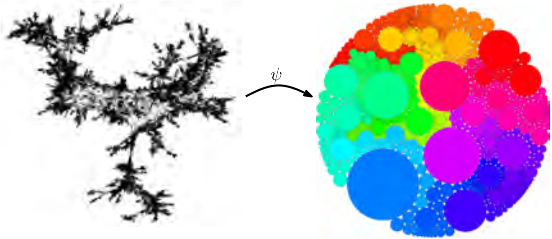
Dancing snake: a natural random walk on the space of discrete "snakes."



1. The dancing snake has a scaling limit called the **Brownian snake**.
2. The x and y coordinates of the Brownian snake's head are two functions.
3. Each of these describes a tree (via the same construction we used to make CRT from Brownian motion).
4. Gluing these two trees together gives a random surface called the **Brownian map**.

Some QLE-based results

- Existence of $\mathrm{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a γ-LQG surface.
- A proof that when $\gamma^2 = 8/3$ and $\eta = 0$, QLE describes the growth of metric balls in Liouville quantum gravity.
- A proof that, under the metric defined by QLE, Liouville quantum gravity is equivalent (as a random metric measure space) to the Brownian map.
- An understanding of a continuum analog of DLA on a random surface corresponding to $\gamma^2 = 2$.



Thanks!

18.177 Universal Random Structures in 2D

Fall 2015