18.175: Lecture 1

Probability spaces and σ-algebras

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Outline

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Distributions on $\mathbb{R}$

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Probability space notation

- **Probability space** is triple $(\Omega, \mathcal{F}, P)$ where Ω is sample space, $\mathcal{F}$ is set of events (the σ-algebra) and $P : \mathcal{F} \to [0, 1]$ is the probability function.
- σ-**algebra** is collection of subsets closed under complementation and countable unions. Call $(\Omega, \mathcal{F})$ a measure space.
- **Measure** is function $\mu : \mathcal{F} \to \mathbb{R}$ satisfying $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{F}$ and countable additivity: $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ for disjoint A_i.
- Measure μ is **probability measure** if $\mu(\Omega) = 1$.

Basic consequences of definitions

- **monotonicity**: $A \subset B$ implies $\mu(A) \leq \mu(B)$
- **subadditivity**: $A \subset \cup_{m=1}^{\infty} A_m$ implies $\mu(A) \leq \sum_{m=1}^{\infty} \mu(A_m)$.
- **continuity from below:** measures of sets A_i in increasing sequence converge to measure of limit $\cup_i A_i$
- **continuity from above:** measures of sets A_i in decreasing sequence converge to measure of intersection $\cap_i A_i$

Why can't σ-algebra be all subsets of Ω?

- Uniform probability measure on $[0,1)$ should satisfy **translation invariance:** If B and a horizontal translation of B are both subsets $[0,1)$, their probabilities should be equal.
- Consider **wrap-around translations** $\tau_r(x) = (x+r) \mod 1$.
- By translation invariance, $\tau_r(B)$ has same probability as B.
- Call x, y "equivalent modulo rationals" if $x - y$ is rational (e.g., $x = \pi - 3$ and $y = \pi - 9/4$). An **equivalence class** is the set of points in $[0,1)$ equivalent to some given point.
- There are uncountably many of these classes.
- Let $A \subset [0,1)$ contain **one** point from each class. For each $x \in [0,1)$, there is **one** $a \in A$ such that $r = x - a$ is rational.
- Then each x in $[0,1)$ lies in $\tau_r(A)$ for **one** rational $r \in [0,1)$.
- Thus $[0,1) = \cup \tau_r(A)$ as r ranges over rationals in $[0,1)$.
- If $P(A) = 0$, then $P(S) = \sum_r P(\tau_r(A)) = 0$. If $P(A) > 0$ then $P(S) = \sum_r P(\tau_r(A)) = \infty$. Contradicts $P(S) = 1$ axiom.

Three ways to get around this

- 1. **Re-examine axioms of mathematics:** the very *existence* of a set A with one element from each equivalence class is consequence of so-called **axiom of choice**. Removing that axiom makes paradox goes away, since one can just suppose (pretend?) these kinds of sets don't exist.
- 2. **Re-examine axioms of probability:** Replace *countable additivity* with *finite additivity*? (Look up Banach-Tarski.)
- 3. **Keep the axiom of choice and countable additivity but don't define probabilities of all sets:** Restrict attention to some σ-algebra of measurable sets.
- Most mainstream probability and analysis takes the third approach. But good to be aware of alternatives (e.g., **axiom of determinacy** which implies that all sets are Lebesgue measurable).

Borel σ-algebra

- The **Borel σ-algebra** $\mathcal{B}$ is the smallest σ-algebra containing all open intervals.
- Say that $\mathcal{B}$ is "generated" by the collection of open intervals.
- Why does this notion make sense? If $\mathcal{F}_i$ are σ-fields (for i in possibly uncountable index set I) does this imply that $\cap_{i \in I} \mathcal{F}_i$ is a σ-field?

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18.175 Theory of Probability
Spring 2014