Lecture 3

MVP + integrable $\Leftrightarrow$ harmonic

Theorem 1 *Suppose $u \in L^1_{loc}$, then u is harmonic $\Leftrightarrow$ u satisfies MVP on Ω.*

Proof: Take C^∞ function ρ on $\mathbb{R}^n$ with properties: (a) $Supp(\rho) \subset \overline{B(0,1)}$; (b) $\rho \geq 0$; (c) ρ is radical, i.e. $\rho(x) = \rho(|x|)$; and (d) $\int_{B(0,1)} \rho(x)dx = 1$.

By these properties, we have

$$1 = \int_B (0,1)\rho(x)dx = \int_0^1 \int_{\partial B(0,s)} \rho(s) d\sigma ds = \int_0^1 \rho(s) n\omega_n s^{n-1} ds.$$

Define $\rho^{(r)}(x) = \frac{1}{r^n}\rho(\frac{|x|}{r}), u_r(x) = \rho^{(r)}(x) * u = \frac{1}{r^n}\int_\Omega \rho(\frac{|x-y|}{r})u(y)dy$. (Without loss of generality, we can assume $u \in L^1(\Omega)$ – otherwise we consider near every point s.t. u is integrable.)

Now we have

$$\begin{aligned}
u_r(y) &= \frac{1}{r^n}\int_\Omega \rho(\frac{|x-y|}{r})u(x)dx \\
&= \frac{1}{r^n}\int_{B(y,r)} \rho(\frac{|x-y|}{r})u(x)dx \\
&= \frac{1}{r^n}\int_0^r \int_{\partial B(y,s)} \rho(\frac{|x-y|}{r})u(x)d\sigma ds \\
&= \frac{1}{r^n}\int_0^r \int_{\partial B(y,s)} \rho(\frac{s}{r})u(x)d\sigma ds \\
&= \frac{1}{r^n}\int_0^r \rho(\frac{s}{r})n\omega_n s^{n-1}u(y)ds \\
&= \frac{n\omega_n u(y)}{r^n}\int_0^r \rho(\frac{s}{r})s^{n-1}ds \\
&= \frac{n\omega_n u(y)}{r^n}\int_0^1 r\rho(t)r^{n-1}t^{n-1}dt \\
&= n\omega_n u(y)\int_0^1 \rho(t)t^{n-1}dt \\
&= u(y).
\end{aligned}$$

But $\rho \in C^\infty \Rightarrow u_r \in C^\infty$, so $u \in C^\infty$. Thus MVP $\Rightarrow$ u is harmonic by last lecture. ■

Weak Solution

For the function $\Gamma(x)$, we have (in distributional sense) $\Delta\Gamma(x) = \delta_0(x)$, i.e. for $\varphi \in C_c^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \Gamma(x)\Delta\varphi(x)dx = \varphi(0) = \int \varphi\delta(0).$$

More generally, $\Delta\Gamma(x-y) = \delta_y(x)$.

Proof: Choose R large so that $Supp\varphi \subset B(0,R)$. Choose ρ small. From Green's formula we get

$$\int_{\mathbb{R}^n - B(0,\rho)} \Gamma\Delta\varphi dx = \int_{\partial B_\rho} (\Gamma\frac{\partial\varphi}{\partial\nu} - u\frac{\partial\Gamma}{\partial\nu})ds.$$

As $\rho \to 0$, we get

$$\int_{\mathbb{R}^n - B(0,\rho)} \Gamma\Delta\varphi dx \to \int_{\mathbb{R}^n} \Gamma\Delta\varphi dx,$$

$$\Gamma(\rho)\int_{\partial B_\rho} \frac{\partial\varphi}{\partial\nu}ds \leq \frac{c}{\rho^{n-2}}\rho^{n-1} \to 0,$$

$$-\int_{\partial B_\rho} u\frac{\partial\Gamma}{\partial\nu} = \frac{1}{n\omega_n}\frac{1}{\rho^{n-1}}\int_{\partial B_\rho} ud\sigma \to u(0),$$

which give what we claimed. ■

Application: $G(x,y) = G(y,x)$

$$\begin{aligned} G(x,y) - G(y,x) &= \int_\Omega (G(x,z)\delta(y-z) - G(y,z)\delta(x-z))dz \\ &= \int_\Omega (G(x,z)\Delta_z\Gamma(y-z) - G(y,z)\Delta_z\Gamma(x-z))dz \\ &= \int_{\partial\Omega} (G(x,z)\frac{\partial}{\partial\nu_z}\Gamma(y-z) - G(y,z)\frac{\partial}{\partial\nu_z}\Gamma(x-z))dz \\ &= 0. \end{aligned}$$

Weyl's Lemma: Regularity of weakly harmonic functions

Theorem 2 *Suppose* $u \in L_0^1(\Omega)$ *satisfies* $\int_\Omega u(x)\Delta\varphi(x)dx = 0$ *for* $\forall\varphi \in C_c^2(\Omega)$. *Then* u *is harmonic in* Ω.

Proof: Without loss of generality, we can assume $u \in L^1(\Omega)$.

Again we take

$$u_r(x) = \rho^{(r)}(x) * u = \frac{1}{r^n}\int_\Omega \rho(\frac{|x-y|}{r})u(y)dy.$$

Claim 1. $\int_\Omega f(y-x)\Delta g(x)dx = \Delta_y \int_\Omega f(y-x)g(x)dx, \forall f, g:$

$$\begin{aligned}\Delta_y \int_\Omega f(y-x)g(x)dx &= \Delta_y \int_\Omega f(x)g(y-z)dz \\ &= \int_\Omega f(z)\Delta_y g(y-z)dz \\ &= \int_\Omega f(y-x)\Delta g(x)dx.\end{aligned}$$

Claim 2. $\int_\Omega u_r(x)\Delta\varphi(x)dx = \int_\Omega u(x)\Delta\varphi_r(x)dx:$

$$\begin{aligned}\int_\Omega u_r(x)\Delta\varphi(x)dx &= \int_\Omega \frac{1}{r^n}(\int_\Omega \rho(\frac{|x-y|}{r})u(y)\Delta\varphi(x)dy)dx \\ &= \int_\Omega (\int_\Omega \frac{1}{r^n}\rho(\frac{|x-y|}{r})u(y)\Delta\varphi(x)dxdy \\ &= \int_\Omega u(y)(\int_\Omega \frac{1}{r^n}\rho(\frac{|x-y|}{r})\Delta\varphi(x)dxdy \\ &= \int_\Omega u(y)\Delta_y(\frac{1}{r^n}\int_\Omega \rho(\frac{|x-y|}{r})\varphi(x)dx)dy \\ &= \int_\Omega u(y)\Delta_y\varphi_r(y)dy.\end{aligned}$$

Claim 3. $u_r(x)$ is harmonic.

In fact, for any $\varphi \in C_c^2(\Omega)$, $\Delta\varphi_r(y) \in C_c^2(\Omega)$, so by the assumption we have

$$\int_\Omega u(y)\Delta_y\varphi_r(y)dy = 0.$$

Thus by claim 2, $\int_\Omega u_r(x)\Delta\varphi(x)dx = 0$ for any $\varphi \in C_c^2(\Omega)$.

But $u_r(x) \in C^\infty(\Omega)$, thus

$$\int_\Omega u_r(x)\Delta\varphi(x)dx = \int_\Omega \Delta u_r(x)\varphi(x)dx.$$

So we get

$$\int_\Omega \Delta u_r(x)\varphi(x)dx = 0, \forall\varphi \in C_c^2(\Omega),$$

which implies $\Delta u_r(x) = 0$, i.e. $u_r(x)$ is harmonic.

Claim 4. $\{u_r\}$ uniquely bounded and equicontinuous on any $\Omega' \subset\subset \Omega$.

In fact, $u_r = \rho^{(r)} * u$ implies

$$\|u_r\|_{L^1} \leq \|\rho^{(r)}\|_{L^1}\|u\|_{L^1} \leq \|u\|_{L^1},$$

so

$$\sup_{\Omega' \subset\subset \Omega} |D^k u_r| \leq C \cdot \sup_{\Omega} |u_r| \leq C\|u\|_{L^1}.$$

Since u_r harmonic, we get

$$u_r(y) = \frac{1}{\omega_n R^n} \int_{B(y,R)} u_r(x) dx,$$

which implies

$$|u_r(y)| \leq \frac{1}{\omega_n R^n} \|u\|_{L^1}.$$

Claim 5. u is smooth.

In fact, by Arzela-Ascoli theorem, there is some subsequence $r_i \to 0, i \to \infty$ s.t. $u_{r_i} \to v \in C^\infty$ on $\Omega' \subset \Omega$.

But $u_{r_i} = \rho^{(r_i)} * u \to u$ in L^1 as $r_i \to 0$, so $u = v$ on Ω'. Thus u is smooth on Ω. Now since u is smooth, we have

$$0 = \int_\Omega u \Delta\varphi = \int_\Omega \varphi \Delta u, \forall \varphi,$$

so $\Delta u = 0$, i.e. u is harmonic. ■