Lecture 0

Course overview

In this course, we will mainly be concerned with the following problems:

1) Harmonic functions $\Delta u = 0$, i.e. $\sum_i u_{ii} = 0$.

Dirichlet problem: $(\Omega \subset \mathbb{R}^n)$

$$\begin{cases} \Delta u = 0 & , \quad x \in \Omega, \\ u = \varphi & , \quad x \in \partial\Omega. \end{cases}$$

2) Heat equation: $u_t = \Delta u, u : \mathbb{R}^{n+1} \to \mathbb{R}$.

Boundary value problem: cylinder domain $\Omega \times [0, T), \Omega \subset \mathbb{R}^n$.

$$\begin{cases} u_t = \Delta u & , \quad (x,t) \in \Omega \times [0,T), \\ u = \varphi & , \quad (x,t) \in \Omega \times \{0\} \cup \partial\Omega \times [0,T). \end{cases}$$

3)Poisson Equation: $(\Omega \subset \mathbb{R}^n)$

$$\begin{cases} \Delta u = f & , \quad x \in \Omega, \\ u = \varphi & , \quad x \in \partial\Omega. \end{cases}$$

For which f, φ, Ω can we solve?

Parabolic:

$$\begin{cases} u_t - \Delta u = f(x,t) & , \quad (x,t) \in \Omega \times [0,T), \\ u = \varphi(x,t) & , \quad (x,t) \in \Omega \times \{0\} \cup \partial\Omega \times [0,T). \end{cases}$$

We will prove existence theorems by method of priori estimates.

For $\Delta u = f$, when does certain regularity of f imply regularity of u?

- If f continuous, is $u \in C^2$? **NO**!

We will always consider in the Hölder spaces $C^\alpha(\Omega), C^{0,\alpha}(\Omega)$. The norm is

$$\|f\|_{C^\alpha(\Omega)} = \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

Thus $f \in C^\alpha(\Omega) \implies |f(x) - f(y)| \leq \|f\|_{C^\alpha(\Omega)} |x-y|^\alpha$.

When $\alpha = 1$, f is just Lipstitz continuous functions.

For $\Delta u = f$ in Ω, we will get Interior Estimates

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|f\|_{C^\alpha(\Omega)} + \|u\|_{C^0(\Omega)}),$$

where $\Omega' \subset\subset \Omega, C = C(\Omega, \Omega')$.

Notion of weak solution:
$\Delta u = f$ **weakly** on Ω if $\int_\Omega u\Delta\varphi = \int_\Omega \varphi f, \forall \varphi \in C_c^2(\Omega)$, here $u \in L^1_{loc}(\Omega)$.

Regularity theorem: If u is a weak solution, then u should has as much regularity as the priori estimates.

In practical problems, it's usually easy to prove existence of weak solutions.

The harder problem: prove weak solution is regular, and therefore solves the original equation strongly.

In general, the global estimates should depend on $\partial\Omega$ and φ:

$$\begin{cases} \Delta u = f & , \quad x \in \Omega, \\ u = \varphi & , \quad x \in \partial\Omega. \end{cases}$$

$f \in C^\alpha(\Omega)$. Assume φ is the restriction of a $C^{2,\alpha}$ function on $\mathbb{R}^n$ to $\partial\Omega$, i.e. φ has a $C^{2,\alpha}$ extension, and $\partial\Omega$ is $C^{2,\alpha}$ smooth. Then $u \in C^{2,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \le C(\|f\|_{C^{2,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)} + \|\varphi\|_{C^{2,\alpha}(\partial\Omega)}).$$

L^p **theory**: $\Delta u = f, f \in L^p(\Omega).((\int_\Omega |f|^p)^{\frac{1}{p}} < \infty)$

If u is a weak solution. Does 2^{nd} order derivation of u belong to L^p, i.e.

$$(\int_\Omega |D^2u|^p)^{\frac{1}{p}} < \infty \ ? \qquad 1 < p < \infty$$

We can get

$$\|u\|_{W^{2,p}(\Omega')} \le C(\|f\|_{L^p} + \|u\|_{L^p}).$$

We just look at Δ. The next is more general elliptic operators:

$$Lu = \sum_{i,j} a^{ij}(x)D_{ij}u + \sum_i b^i(x)D_iu + c(x)u = f.$$

We also consider the following problems:

$$\begin{cases} Lu = f & , \quad x \in \Omega, \\ u = \varphi & , \quad x \in \partial\Omega. \end{cases}$$

$$\begin{cases} u_t - Lu = f(x,t) & , \quad (x,t) \in \Omega \times [0,T), \\ u = \varphi(x,t) & , \quad (x,t) \in \Omega \times \{0\} \cup \partial\Omega \times [0,T). \end{cases}$$

We call L is uniformly elliptic if $\Lambda I \geq (a^{ij}) \geq \lambda I, \lambda > 0$.

Schauder Theory: L is uniformly elliptic, $a^{ij}, b_i, c \in C^\alpha(\Omega)$, then

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|f\|_{C^\alpha(\Omega)} + \|u\|_{C^0(\Omega)}).$$

Idea: Assuming coefficients are all C^α, locally L is close to a constant coefficients operator.

Maximum principle: Bound C^0 norm of solution in terms of boundary data of f.

$$\|u\|_{C^0(\Omega)} \leq C(\|f\|_{C^0(\Omega)} + \sup |\varphi|).$$

This is an A Priori estimate:
I) Assume solution exists;
II) Prove solutions satisfies a priori bounds;
III) Therefore the solution exists.

Motivation: If you want to completely understand Perelman's proof of Poincaré conjecture, you have to know this stuff.

$$\frac{\partial}{\partial t} g = -2Ric$$
$$\frac{\partial}{\partial t} g_{ij} \sim \Delta_g g_{ij} + lower\ terms.$$

Fundamental Result: (M^3, g) compact 3-manifold, then $\exists \varepsilon > 0$ s.t. Ricci flow system has a smooth solution on $M \times [0, \varepsilon)$.
(This is called short time existence theorem.)

Examples of harmonic functions in $\mathbb{R}^n$

a) Constant.
b) linear functions.
c) Homogeneous harmonic polynomials: $\mathcal{H}^k(\mathbb{R}^n)$.
$dim \mathcal{H}^k(\mathbb{R}^n) = (2k + n - 2)\frac{(k+n-3)!}{k!(n-2)!}$.
d) $n = 2$, the real or image part of holomorphic functions is harmonic.
They are C^∞. Even more, they are C^ω.
e) Fundamental solution

$$u(x) = \begin{cases} \frac{C}{r^{n-2}} & , \quad n > 2, \\ C \ln r & , \quad n = 2. \end{cases}$$

is harmonic on $\mathbb{R}^n - \{0\}$.

Fundamental solutions for Laplacian and heat operator

Definition 1

$$\Gamma(x,y) = \begin{cases} \frac{1}{n(2-n)\omega_n}|x-y|^{2-n} & , \quad n > 2, \\ \frac{1}{2\pi}\log|x-y| & , \quad n = 2. \end{cases}$$

Γ is harmonic:

$$\frac{\partial\Gamma(x,y)}{\partial x^i} = \frac{1}{n\omega_n}(x^i - y^i)\frac{1}{|x-y|^n}$$

$$\implies \frac{\partial^2\Gamma(x,y)}{\partial x^i \partial x^j} = \frac{1}{n\omega_n}\{\frac{1}{|x-y|^n}\delta_{ij} - \frac{n(x^i-y^i)(x^j-y^j)}{|x-y|^{n+2}}\}$$

$$\implies \Delta_x\Gamma(x,y) = 0$$

$$\implies \Delta_y\Gamma(x,y) = 0.$$

Definition 2

$$\Lambda(x,y,t,t_0) = \frac{1}{(4\pi|t-t_0|)^{n/2}}e^{\frac{|x-y|^2}{4(t_0-t)}}.$$

We have $\Lambda_t = \Delta\Lambda$:

$$\Lambda_t = -\frac{n}{2}\frac{1}{(t-t_0)}\Lambda + \frac{|x-y|^2}{4(t-t_0)^2}\Lambda$$

$$\Lambda_{x^i} = \frac{x^i - y^i}{2(t_0-t)}\Lambda$$

$$\implies \Lambda_{x^ix^i} = \frac{(x^i-y^i)^2}{4(t_0-t)}\Lambda + \frac{1}{2(t_0-t)}\Lambda$$

$$\implies \Delta\Lambda = -\frac{n}{2}\frac{1}{(t-t_0)}\Lambda + \frac{|x-y|^2}{4(t-t_0)^2}\Lambda.$$

Heat Kernel:

$$K(x,y,t) = \frac{1}{(4\pi t)^{n/2}}e^{\frac{-|x-y|^2}{4t}}.$$

It's easy to check $K_t = \Delta K$.

Suppose $u : \mathbb{R}^n \mapsto \mathbb{R}$ is bounded and C^0, then

$$\overline{u}(x,t) = \int_{\mathbb{R}^n} K(x,y,t)u(y)dy$$

is C^∞ on $\mathbb{R}^n \times (0,\infty)$ and $\overline{u}_t = \Delta\overline{u}$.