6. Test functions

So far we have largely been dealing with integration. One thing we have seen is that, by considering dual spaces, we can think of functions as functionals. Let me briefly review this idea.

Consider the unit ball in $\mathbb{R}^n$,

$$\overline{\mathbb{B}}^n = \{x \in \mathbb{R}^n\,;\, |x| \le 1\}\ .$$

I take the *closed* unit ball because I want to deal with a compact metric space. We have dealt with several Banach spaces of functions on $\overline{\mathbb{B}^n}$, for example

$$C(\overline{\mathbb{B}^n}) = \big\{u : \overline{\mathbb{B}^n} \longrightarrow \mathbb{C}\,;\, u \text{ continuous}\big\}$$
$$L^2(\overline{\mathbb{B}^n}) = \left\{u : \overline{\mathbb{B}^n} \longrightarrow \mathbb{C}; \text{Borel measurable with } \int |u|^2\, dx < \infty\right\}.$$

Here, as always below, dx is Lebesgue measure and functions are identified if they are equal almost everywhere.

Since $\overline{\mathbb{B}^n}$ is compact we have a natural inclusion

$$C(\overline{\mathbb{B}^n}) \hookrightarrow L^2(\overline{\mathbb{B}^n})\,. \tag{6.1}$$

This is also a topological inclusion, i.e., is a bounded linear map, since

$$\|u\|_{L^2} \le C\|u\|_\infty \tag{6.2}$$

where C^2 is the volume of the unit ball.

In general if we have such a set up then

Lemma 6.1. *If $V \hookrightarrow U$ is a subspace with a stronger norm,*

$$\|\varphi\|_U \le C\|\varphi\|_V\ \forall\ \varphi \in V$$

then restriction gives a continuous linear map

$$U' \to V',\ U' \ni L \longmapsto \tilde L = L|_V \in V',\ \|\tilde L\|_{V'} \le C\|L\|_{U'}\,. \tag{6.3}$$

If V is dense in U then the map (6.3) *is injective.*

Proof. By definition of the dual norm

$$\begin{aligned}\|\tilde L\|_{V'} &= \sup\left\{\left|\tilde L(v)\right|\ ;\ \|v\|_V \le 1\,,\ v \in V\right\}\\ &\le \sup\left\{\left|\tilde L(v)\right|\ ;\ \|v\|_U \le C\,,\ v \in V\right\}\\ &\le \sup\left\{|L(u)|\ ;\|u\|_U \le C\,,\ u \in U\right\}\\ &= C\|L\|_{U'}\,.\end{aligned}$$

If $V \subset U$ is dense then the vanishing of $L : U \longrightarrow \mathbb{C}$ on V implies its vanishing on U.

$\square$

Going back to the particular case (6.1) we do indeed get a continuous map between the dual spaces

$$L^2(\overline{\mathbb{B}^n}) \cong (L^2(\overline{\mathbb{B}^n}))' \to (C(\overline{\mathbb{B}^n}))' = M(\overline{\mathbb{B}^n})\,.$$

Here we use the Riesz representation theorem and duality for Hilbert spaces. The map use here is supposed to be *linear* not antilinear, i.e.,

$$L^2(\overline{\mathbb{B}^n}) \ni g \longmapsto \int \cdot g\, dx \in (C(\overline{\mathbb{B}^n}))'\,. \tag{6.4}$$

So the idea is to make the space of 'test functions' as small as reasonably possible, while still retaining *density* in reasonable spaces.

Recall that a function $u : \mathbb{R}^n \to \mathbb{C}$ is *differentiable* at $\overline{x} \in \mathbb{R}^n$ if there exists $a \in \mathbb{C}^n$ such that

$$|u(x) - u(\overline{x}) - a \cdot (x - \overline{x})| = o(|x - \overline{x}|)\,. \tag{6.5}$$

The 'little oh' notation here means that given $\epsilon > 0$ there exists $\delta > 0$ s.t.

$$|x - \overline{x}| < \delta \Rightarrow |u(x) - u(\overline{x}) - a(x - \overline{x})| < \epsilon\,|x - \overline{x}|\,.$$

The coefficients of $a = (a_1, \dots, a_n)$ are the partial derivations of u at $\overline{x}$,

$$a_i = \frac{\partial u}{\partial x_j}(\overline{x})$$

since

$$a_i = \lim_{t \to 0} \frac{u(\overline{x} + te_i) - u(\overline{x})}{t}\,, \tag{6.6}$$

$e_i = (0, \dots, 1, 0, \dots, 0)$ being the ith basis vector. The function u is said to be *continuously differentiable* on $\mathbb{R}^n$ if it is differentiable at *each* point $\overline{x} \in \mathbb{R}^n$ and each of the n partial derivatives are continuous,

$$\frac{\partial u}{\partial x_j} : \mathbb{R}^n \to \mathbb{C}\,. \tag{6.7}$$

Definition 6.2. *Let $\mathcal{C}^1_0(\mathbb{R}^n)$ be the subspace of $\mathcal{C}_0(\mathbb{R}^n) = \mathcal{C}^0_0(\mathbb{R}^n)$ such that each element $u \in \mathcal{C}^1_0(\mathbb{R}^n)$ is continuously differentiable and $\frac{\partial u}{\partial x_j} \in \mathcal{C}_0(\mathbb{R}^n)$, $j = 1, \dots, n$.*

Proposition 6.3. *The function*

$$\|u\|_{\mathcal{C}^1} = \|u\|_\infty + \sum_{i=1}^n \|\frac{\partial u}{\partial x_1}\|_\infty$$

is a norm on $\mathcal{C}^1_0(\mathbb{R}^n)$ with respect to which it is a Banach space.

Proof. That $\| \ \|_{\mathcal{C}^1}$ is a norm follows from the properties of $\| \ \|_\infty$. Namely $\|u\|_{\mathcal{C}^1} = 0$ certainly implies $u = 0$, $\|au\|_{\mathcal{C}^1} = |a| \ \|u\|_{\mathcal{C}^1}$ and the triangle inequality follows from the same inequality for $\| \ \|_\infty$.

Similarly, the main part of the completeness of $\mathcal{C}^1_0(\mathbb{R}^n)$ follows from the completeness of $\mathcal{C}^0_0(\mathbb{R}^n)$. If $\{u_n\}$ is a Cauchy sequence in $\mathcal{C}^1_0(\mathbb{R}^n)$ then u_n and the $\frac{\partial u_n}{\partial x_j}$ are Cauchy in $\mathcal{C}^0_0(\mathbb{R}^n)$. It follows that there are limits of these sequences,

$$u_n \to v\,,\ \frac{\partial u_n}{\partial x_j} \to v_j \in \mathcal{C}^0_0(\mathbb{R}^n)\,.$$

However we do have to check that v is continuously differentiable and that $\frac{\partial v}{\partial x_j} = v_j$.

One way to do this is to use the Fundamental Theorem of Calculus in each variable. Thus

$$u_n(\overline{x} + te_i) = \int_0^t \frac{\partial u_n}{\partial x_j}(\overline{x} + se_i)\,ds + u_n(\overline{x})\,.$$

As $n \to \infty$ all terms converge and so, by the continuity of the integral,

$$u(\overline{x} + te_i) = \int_0^t v_j(\overline{x} + se_i)\,ds + u(\overline{x})\,.$$

This shows that the limit in (6.6) exists, so $v_i(\overline{x})$ is the partial derivation of u with respect to x_i. It remains only to show that u is indeed differentiable at each point and I leave this to you in Problem 17.

□

So, almost by definition, we have an example of Lemma 6.1,

$$\mathcal{C}^1_0(\mathbb{R}^n) \hookrightarrow \mathcal{C}^0_0(\mathbb{R}^n).$$

It is in fact dense but I will not bother showing this (yet). So we know that

$$(\mathcal{C}^0_0(\mathbb{R}^n))' \to (\mathcal{C}^1_0(\mathbb{R}^n))'$$

and we expect it to be injective. Thus there are *more* functionals on $\mathcal{C}^1_0(\mathbb{R}^n)$ including things that are 'more singular than measures'.

An example is related to the Dirac delta

$$\delta(\overline{x})(u) = u(\overline{x})\,,\ u \in \mathcal{C}^0_0(\mathbb{R}^n)\,,$$

namely

$$\mathcal{C}^1_0(\mathbb{R}^n) \ni u \longmapsto \frac{\partial u}{\partial x_j}(\overline{x}) \in \mathbb{C}\,.$$

This is clearly a continuous linear functional which it is only just to denote $\frac{\partial}{\partial x_j}\delta(\overline{x})$.

Of course, why stop at one derivative?

Definition 6.4. *The space* $\mathcal{C}_0^k(\mathbb{R}^n) \subset \mathcal{C}_0^1(\mathbb{R}^n)\; k \ge 1$ *is defined inductively by requiring that*

$$\frac{\partial u}{\partial x_j} \in \mathcal{C}_0^{k-1}(\mathbb{R}^n)\,,\ j = 1, \dots, n\,.$$

The norm on $\mathcal{C}_0^k(\mathbb{R}^n)$ *is taken to be*

$$\|u\|_{\mathcal{C}^k} = \|u\|_{\mathcal{C}^{k-1}} + \sum_{j=1}^n \|\frac{\partial u}{\partial x_j}\|_{\mathcal{C}^{k-1}}\,. \tag{6.8}$$

These are all Banach spaces, since if $\{u_n\}$ is Cauchy in $\mathcal{C}_0^k(\mathbb{R}^n)$, it is Cauchy and hence convergent in $\mathcal{C}_0^{k-1}(\mathbb{R}^n)$, as is $\partial u_n/\partial x_j$, $j = 1, \dots, n-1$. Furthermore the limits of the $\partial u_n/\partial x_j$ are the derivatives of the limits by Proposition 6.3.

This gives us a sequence of spaces getting 'smoother and smoother'

$$\mathcal{C}_0^0(\mathbb{R}^n) \supset \mathcal{C}_0^1(\mathbb{R}^n) \supset \cdots \supset \mathcal{C}_0^k(\mathbb{R}^n) \supset \cdots\,,$$

with norms getting larger and larger. The duals can also be expected to get larger and larger as k increases.

As well as looking at functions getting smoother and smoother, we need to think about 'infinity', since $\mathbb{R}^n$ is not compact. Observe that an element $g \in L^1(\mathbb{R}^n)$ (with respect to Lebesgue measure by default) defines a functional on $\mathcal{C}_0^0(\mathbb{R}^n)$ — and hence *all* the $\mathcal{C}_0^k(\mathbb{R}^n)$s. However a function such as the constant function 1 is *not* integrable on $\mathbb{R}^n$. Since we certainly want to talk about this, and polynomials, we consider a second condition of *smallness at infinity.* Let us set

$$\langle x \rangle = (1 + |x|^2)^{1/2} \tag{6.9}$$

a function which is the size of $|x|$ for $|x|$ large, but has the virtue of being smooth[10]

Definition 6.5. *For any* $k,\ l \in \mathbb{N} = \{1, 2, \cdots\}$ *set*

$$\langle x \rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n) = \left\{u \in \mathcal{C}_0^k(\mathbb{R}^n)\,;\, u = \langle x \rangle^{-l} v\,,\ v \in \mathcal{C}_0^k(\mathbb{R}^n)\right\}\,,$$

with norm, $\|u\|_{k,l} = \|v\|_{\mathcal{C}^k}\,,\ v = \langle x \rangle^l u.$

Notice that the definition just says that $u = \langle x \rangle^{-l} v$, with $v \in \mathcal{C}_0^k(\mathbb{R}^n)$. It follows immediately that $\langle x \rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n)$ is a Banach space with this norm.

Definition 6.6. *Schwartz' space*[11] *of test functions on* $\mathbb{R}^n$ *is*

$$\mathcal{S}(\mathbb{R}^n) = \left\{u : \mathbb{R}^n \to \mathbb{C}; u \in \langle x \rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n) \text{ for all } k \text{ and } l \in \mathbb{N}\right\}.$$

[10] See Problem 18.

[11] Laurent Schwartz – this one with a 't'.

It is not immediately apparent that this space is non-empty (well 0 is in there but...); that

$$\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$$

is Problem 19. There are lots of other functions in there as we shall see.

Schwartz' idea is that the dual of $\mathcal{S}(\mathbb{R}^n)$ should contain all the 'interesting' objects, at least those of 'polynomial growth'. The problem is that we do *not* have a good norm on $\mathcal{S}(\mathbb{R}^n)$. Rather we have a *lot* of them. Observe that

$$\langle x\rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n) \subset \langle x\rangle^{-l'}\mathcal{C}_0^{k'}(\mathbb{R}^n) \text{ if } l \geq l' \text{ and } k \geq k'\,.$$

Thus we see that as a linear space

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_k \langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n). \tag{6.10}$$

Since these spaces are getting smaller, we have a countably infinite number of norms. For this reason $\mathcal{S}(\mathbb{R}^n)$ is called a *countably normed* space.

Proposition 6.7. *For $u \in \mathcal{S}(\mathbb{R}^n)$, set*

$$\|u\|_{(k)} = \|\langle x\rangle^k u\|_{\mathcal{C}^k} \tag{6.11}$$

and define

$$d(u,v) = \sum_{k=0}^{\infty} 2^{-k}\frac{\|u-v\|_{(k)}}{1+\|u-v\|_{(k)}}, \tag{6.12}$$

then d is a distance function in $\mathcal{S}(\mathbb{R}^n)$ with respect to which it is a complete metric space.

Proof. The series in (6.12) certainly converges, since

$$\frac{\|u-v\|_{(k)}}{1+\|u-v\|_{(k)}} \leq 1.$$

The first two conditions on a metric are clear,

$$d(u,v) = 0 \Rightarrow \|u-v\|_{\mathcal{C}_0} = 0 \Rightarrow u = v,$$

and symmetry is immediate. The triangle inequality is perhaps more mysterious!

Certainly it is enough to show that

$$\tilde{d}(u,v) = \frac{\|u-v\|}{1+\|u-v\|} \tag{6.13}$$

is a metric on any normed space, since then we may sum over k. Thus we consider

$$\frac{\|u-v\|}{1+\|u-v\|}+\frac{\|v-w\|}{1+\|v-w\|} \\ =\frac{\|u-v\|(1+\|v-w\|)+\|v-w\|(1+\|u-v\|)}{(1+\|u-v\|)(1+\|v-w\|)}.$$

Comparing this to $\tilde{d}(v,w)$ we must show that

$$(1+\|u-v\|)(1+\|v-w\|)\|u-w\| \\ \le (\|u-v\|(1+\|v-w\|)+\|v-w\|(1+\|u-v\|))(1+\|u-w\|).$$

Starting from the LHS and using the triangle inequality,

$$\begin{aligned}\text{LHS} &\le \|u-w\|+(\|u-v\|+\|v-w\|+\|u-v\|\|v-w\|)\|u-w\| \\ &\le (\|u-v\|+\|v-w\|+\|u-v\|\|v-w\|)(1+\|u-w\|) \\ &\le \text{RHS}.\end{aligned}$$

Thus, d is a metric.

Suppose u_n is a Cauchy sequence. Thus, $d(u_n,u_m)\to 0$ as $n,m\to\infty$. In particular, given

$$\epsilon>0\ \exists\ N \text{ s.t. } n,m>N \text{ implies} \\ d(u_n,u_m)<\epsilon 2^{-k}\ \forall\ n,m>N.$$

The terms in (6.12) are all positive, so this implies

$$\frac{\|u_n-u_m\|_{(k)}}{1+\|u_n-u_m\|_{(k)}}<\epsilon\ \forall\ n,m>N.$$

If $\epsilon<1/2$ this in turn implies that

$$\|u_n-u_m\|_{(k)}<2\epsilon,$$

so the sequence is Cauchy in $\langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n)$ for each k. From the completeness of these spaces it follows that $u_n\to u$ in $\langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n)_j$ for each k. Given $\epsilon>0$ choose k so large that $2^{-k}<\epsilon/2$. Then $\exists\ N$ s.t. $n>N$

$$\Rightarrow \|u-u_n\|_{(j)}<\epsilon/2\ n>N,\ j\le k.$$

Hence

$$d(u_n,u)=\sum_{j\le k}2^{-j}\frac{\|u-u_n\|_{(j)}}{1+\|u-u_n\|_{(j)}}$$
$$+\sum_{j>k}2^{-j}\frac{\|u-u_n\|_{(j)}}{1+\|u-u_n\|_{(j)}}$$
$$\le \epsilon/4+2^{-k}<\epsilon.$$

This $u_n\to u$ in $\mathcal{S}(\mathbb{R}^n)$. □

As well as the Schwartz space, $\mathcal{S}(\mathbb{R}^n)$, of functions of rapid decrease with all derivatives, there is a smaller 'standard' space of test functions, namely

$$\mathcal{C}^\infty_c(\mathbb{R}^n)=\{u\in\mathcal{S}(\mathbb{R}^n);\operatorname{supp}(u)\Subset\mathbb{R}^n\}\,, \tag{6.14}$$

the space of smooth functions of compact support. Again, it is not quite obvious that this has any non-trivial elements, but it does as we shall see. If we fix a compact subset of $\mathbb{R}^n$ and look at functions with support in that set, for instance the closed ball of radius $R>0$, then we get a closed subspace of $\mathcal{S}(\mathbb{R}^n)$, hence a complete metric space. One 'problem' with $\mathcal{C}^\infty_c(\mathbb{R}^n)$ is that it does not have a complete metric topology which restricts to this topology on the subsets. Rather we must use an *inductive limit* procedure to get a decent topology.

Just to show that this is not really hard, I will discuss it briefly here, but it is not used in the sequel. In particular I will not do this in the lectures themselves. By definition our space $\mathcal{C}^\infty_c(\mathbb{R}^n)$ (denoted traditionally as $\mathcal{D}(\mathbb{R}^n)$) is a countable union of subspaces

$$\mathcal{C}^\infty_c(\mathbb{R}^n)=\bigcup_{n\in\mathbb{N}}\dot{\mathcal{C}}^\infty_c(B(n)),\ \dot{\mathcal{C}}^\infty_c(B(n))=\{u\in\mathcal{S}(\mathbb{R}^n);u=0\text{ in }|x|>n\}. \tag{6.15}$$

Consider

$$\mathcal{T}=\{U\subset\mathcal{C}^\infty_c(\mathbb{R}^n);U\cap\dot{\mathcal{C}}^\infty_c(B(n))\text{ is open in }\dot{\mathcal{C}}^\infty(B(n))\text{ for each }n\}. \tag{6.16}$$

This is a topology on $\mathcal{C}^\infty_c(\mathbb{R}^n)$ – contains the empty set and the whole space and is closed under finite intersections and arbitrary unions – simply because the same is true for the open sets in $\dot{\mathcal{C}}^\infty(B(n))$ for each n. This is in fact the inductive limit topology. One obvious question is:- what does it mean for a linear functional $u:\mathcal{C}^\infty_c(\mathbb{R}^n)\longrightarrow\mathbb{C}$ to be continuous? This just means that $u^{-1}(O)$ is open for each open set in $\mathbb{C}$. Directly from the definition this in turn means that $u^{-1}(O)\cap\dot{\mathcal{C}}^\infty(B(n))$

should be open in $\dot{\mathcal{C}}^\infty(B(n))$ for each n. This however just means that, restricted to each of these subspaces u is continuous. If you now go forwards to Lemma 7.3 you can see what this means; see Problem 74.

Of course there is a lot more to be said about these spaces; you can find plenty of it in the references.