Lecture 9

We have a manifold $\mathbb{C}P^n$. Take

$$P(z_0, \ldots, Pz_n) = \sum_{|\alpha|=m} c_\alpha z^\alpha$$

a homogenous polynomial. Then

1. $P(\lambda z) = \lambda^m P(z)$, so if $P(z) = 0$ then $P(\lambda z) = 0$
2. Euler's identity holds

$$\sum_{i=0}^{n} z_i \frac{\partial P}{\partial z_i} = mP$$

Lemma. *The following are equivalent*

1. *For all $z \in \mathbb{C}^{n+1} - \{0\}$, $dP_z \neq 0$*
2. *For all $z \in \mathbb{C}^{n+1} - \{0\}$, $P(z) = 0$, $dP_z \neq 0$.*

we call P ***non-singular*** *if one of these holds.*

If $X = \{[z_0, \ldots, z_n], P(z) = 0\}$. Note that this is a well-defined property of homogeneous polynomials.

Theorem. *If P is non-singular, X s an $n-1$ dimensional submanifold of $\mathbb{C}P^n$.*

Proof. Let $U_0, \ldots, U_n$ be the standard atlas for $\mathbb{C}P^n$. It is enough to check that $X \cap U_i$ is a submanifold of U_i. WE'll check this for $i = 0$.

Consider the map $\gamma \mathbb{C}^n \xrightarrow{\cong} U_0$ given by

$$\gamma(z_1, \ldots, z_n) = [1, z_1, \ldots, z_n]$$

It is enough to show that $X_0 = \gamma^{-1}(X)$ is a complex $n-1$ dimensional submanifold of $\mathbb{C}^n$. Let $p(z_1, \ldots, z_n) = P(1, z_1, \ldots, z_n)$. X_0 is the set of all points such that $p = 0$. It is enough to show that $p(z) = 0$ implies $dp_z \neq 0$ (showed last time that this would then define a submanifold)

Suppose $dp(z) = p(z) = 0$. Then

$$p(1, z_1, \ldots, z_n) = 0 = \frac{\partial P}{\partial z_i}(1, z_1, \ldots, z_n) = 0 \qquad i = 1, \ldots, n$$

By the Euler Identity

$$0 = P(1, z_1, \ldots, z_n) = \sum_{i=0}^{n} z_i \frac{\partial P}{\partial z_i}(1, z_1, \ldots, z_n) + \sum \frac{\partial P}{\partial z_i}(1, z_1, \ldots, z_n)$$

So $\frac{\partial P}{\partial z_i}(1, z_1, \ldots, z_n) = 0$, which is a contradiction because we assumed $p \neq 0$. $\square$

Theorem (Uniqueness of Analytic Continuation). *X a connected complex manifold, $V \subseteq X$ is an open set , $f, g \in \mathcal{O}(X)$. If $f = g$ on V then $f = g$ on all of X.*

Sketch. Local version of UAC plus the following connectedness lemma ☐

Lemma. *For $p, q \in X$ there exists open sets U_i, $i = 1, \ldots, n$ such that*

1. *U_i is biholomorphic to a connected open subset of $\mathbb{C}^n$*
2. *$p \in U_1$*
3. *$q \in U_n$*
4. *$U_i \cap U_{i+1} \neq \emptyset$.*

Theorem. *If X is a connected complex manifold and $f \in \mathcal{O}(X)$ then if for some $p \in X$, $|f| : X \to \mathbb{R}$ takes a local maximum then f is constant.*

Corollary. *If X is compact and connected $\mathcal{O}(X) = \mathbb{C}$.*

This implies that the Whitney embedding theorem does not hold for holomorphic manifolds.

Let X be a complex n-dimensional manifold, X a real $2n$ dimensional manifold. Then if $p \in X$ then T_pX is a real 2n-dimensional vector space and T_pX is a complex n-dimensional vector space.

Think for the moment of T_pX as being a $2n$-dimensional $\mathbb{R}$-linear vector space. Define

$$J_p : T_pX \to T_pX \qquad J_pv = \sqrt{-1}v$$

J_p is $\mathbb{R}$-linear map with the property that $J_p^2 = -I$. We want to find the eigenvectors. First take $T_p \otimes \mathbb{C}$ and extend J_p to this by

$$J_p(v \otimes c) = J_pv \otimes c$$

Now, J_p is $\mathbb{C}$-linear, $J_p : T_p \otimes \mathbb{C} \to T_p \otimes \mathbb{C}$. Also, we can introduce a complex conjugation operator

$$: T_p \otimes \mathbb{C} \to T_p \otimes \mathbb{C} \qquad v \otimes c \mapsto v \otimes \bar{c}$$

We can split the tangent space by

$$T_p \otimes \mathbb{C} = T_p^{1,0} \oplus T_p^{0,1}$$

where $v \in T_p^{1,0}$ if $J_pv = +\sqrt{-1}v$ and $v \in T_p^{0,1}$ if $J_pv = -\sqrt{-1}v$. i.e. we break $T_p \otimes \mathbb{C}$ into eigenspaces.

If $v \in T_p^{1,0}$ iff $\bar{v} \in T_p^{0,1}$ and so the dimension of the two parts of the tangent spaces are equal.

We can also take $T_p^* \otimes \mathbb{C} = (T_p^*)^{1,0} \oplus (T_p^*)^{0,1}$ and $l \in (T_p^*)^{1,0}$ if and only if $J_p^*l = \sqrt{-1}l$, $l \in (T_p^*)^{0,1}$ if $J_p^*l = -\sqrt{-1}l$.

Check that $l \in (T_p^*)^{1,0}$ if and only if $l : T_p \to \mathbb{C}$ is actually $\mathbb{C}$-linear. To do this $J^*l = \sqrt{-1}l$ implies $J_p^*l(v) = l(J_pv) = \sqrt{-1}l(v)$ which implies that l is $\mathbb{C}$-linear.

Corollary. *U is open in X and $p \in U$. Then if $f \in \mathcal{O}(U$ then $df_p \in (T_p^*)^{1,0}$.*

Corollary. *$(U, z_1, \ldots, z_n)$ a coordinate patch then $(dz_1)_p, \ldots, (dz_n)_p$ is a basis of $(T_p^*)^{1,0}$ and $(d\bar{z}_1)_p, \ldots, (d\bar{z}_n)_p$ is a basis of $(T_p^*)^{0,1}$.*

From the splitting above we get a splitting of the exterior product

$$\Lambda^k(T_p^* \otimes \mathbb{C}) = \bigoplus_{l+m=k} \Lambda^{l,m}(T_p^* \otimes \mathbb{C})$$

for $\nu_1, \ldots, \nu_n$ a basis of $T_p^* \otimes \mathbb{C}$ then

$$\omega \in \Lambda^{l,m}(T_p^* \otimes \mathbb{C}) \Leftrightarrow \omega = \sum c_{I,J} \nu_I \wedge \bar{\nu}_J$$

We also get a splitting in the tangent bundle

$$\Lambda^k(T^* \otimes \mathbb{C}) = \bigoplus_{l+m=k} \Lambda^{k,l}(T^* \otimes \mathbb{C})$$

since $\Omega^k(X)$ is sections of $\Lambda^k(T^* \otimes \mathbb{C})$. Then

$$\Omega^k(X) = \bigoplus_{l+m=k} \Lambda^{l,m}(X)$$

Locally when $(U, z_1, \ldots, z_n)$ is a coordinate patch, $\omega \in \Omega^{l,m}(U)$ iff

$$\omega = \sum a_{I,J} dz_I \wedge d\bar{z}_J$$

so we've extended the Dolbeault complex to arbitrary manifolds.