18.117 Problem Set #3

1. Let V be the volume of X. Show that if

$$\max |K(x,y)| < \frac{\epsilon}{V}, \qquad 0 < \epsilon < 1$$

then $I - T_K$ is invertible and its inverse is of the form $I - T_L$, $L \in C^\infty(X \times X)$.

2. T_K is a finite rank smoothing operator if K is of the form

$$K(x,y) = \sum_{i=1}^{N} f_i(x) g_i(y)$$

   (a) Show that if T_K is a finite rank smoothing operator and T_L is any smoothing operator then $T_K T_L$ and $T_L T_K$ are finite ranks smoothing operators.

   (b) Show that if T_K is a finite rank smoothing oeprator then the operator $I - T_K$ has finite dimensional kernel and co-kernel.

3. Show that for every $K \in C^\infty(X \times X)$ and every $\epsilon > 0$ there exists a function, $K_1 \in C^\infty(X \times X)$ of the form above such that

$$\sup |K \overset{1}{-} K_1|(x,y) < \epsilon$$

4. Prove that if T_K is a smoothing operator, then the operator $I - T_K : C^\infty(X) \to C^\infty(X)$ has finite dimensional kernel and cokernel.

5. Prove that the operator $I - T_K$ has a finite dimensional kernel and $f \in C^\infty(X)$ is in the image of this operator if and only if it is orthogonal to the kernel of the operator $I - T_L$, where $L(x,y) = \overline{K(x,y)}$.