18.112 Functions of a Complex Variable
Fall 2008

Solution for 18.112 ps 5

1(Prob 1(f) on P161).

Solution: The function

$$f(z) = \frac{1}{z^m(1-z)^n}$$

has two poles, 0 is a pole of order m and 1 is a pole of order n. At these poles, we have the following expansions via Taylor series

$$\begin{aligned} f(z) &= \frac{1}{z^m}\left[1 + nz + \frac{n(n+1)}{2!}z^2 + \cdots + \frac{n(n+1)\cdots(n+m-2)}{(m-1)!}z^{m-1} + \varphi_m(z)z^m\right] \\ &= \cdots + \binom{n+m-2}{m-1}\frac{1}{z} + \cdots, \end{aligned}$$

thus

$$\mathrm{Res}_{z=0} f(z) = \binom{n+m-2}{m-1}.$$

By the symmetry

$$m \longleftrightarrow n,\ z \longleftrightarrow 1-z,$$

we get immediately

$$\begin{aligned} f(z) &= \cdots + \binom{m+n-2}{n-1}\frac{1}{1-z} + \cdots \\ &= \cdots - \binom{m+n-2}{m-1}\frac{1}{z-1} + \cdots \end{aligned},$$

which implies

$$\mathrm{Res}_{z=1} f(z) = -\binom{n+m-2}{m-1}.$$

2(Prob 3(b) on P161).

Solution: According to (2) on page 156, we know that

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{x^4 + 5x^2 + 6} = 2\pi i \sum_{y>0} \text{Res}(\frac{z^2}{z^4 + 5z^2 + 6}).$$

Now

$$\begin{aligned} f(z) &= \frac{z^2}{z^4 + 5z^2 + 6} \\ &= \frac{z^2}{(z^2+2)(z^2+3)} \\ &= \frac{z^2}{(z-\sqrt{3}i)(z+\sqrt{3}i)(z-\sqrt{2}i)(z+\sqrt{2}i)}, \end{aligned}$$

which has only simple poles, and

$$\begin{aligned} \sum_{y>0} \text{Res} f(z) &= \text{Res}_{z=\sqrt{3}i} f(z) + \text{Res}_{z=\sqrt{2}i} f(z) \\ &= \frac{\sqrt{3}}{2i} - \frac{\sqrt{2}}{2i}. \end{aligned}$$

Since the integrand is even function, we have

$$\begin{aligned} \int_{0}^{\infty} \frac{x^2 dx}{x^4 + 5x^2 + 6} &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2 dx}{x^4 + 5x^2 + 6} \\ &= \frac{\pi}{2}(\sqrt{3} - \sqrt{2}). \end{aligned}$$

3(Prob 3(f) on P161).

Solution: Suppose $a \neq 0$. By (3) on page 156, we know that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \text{Im} \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2} dx$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2} dx &= 2\pi i \sum_{y>0} \text{Res}(\frac{ze^{iz}}{z^2 + a^2}) \\ &= 2\pi i \text{Res}_{z=i|a|}(\frac{ze^{iz}}{z^2 + a^2}) \\ &= 2\pi i \cdot \frac{1}{2} e^{-|a|} \\ &= \pi i e^{-|a|}. \end{aligned}$$

Since the integrand is even function,

$$\int_0^\infty \frac{x\sin x}{x^2+a^2}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x}{x^2+a^2}dx$$
$$= \frac{1}{2}\mathrm{Im}(\pi i e^{-|a|})$$
$$= \frac{\pi}{2}e^{-|a|}.$$

In the case $a = 0$, the result is the same (See page 158).

4(Prob 3(h) on P161).

Solution: Define $\log z$ to be single-valued on $\mathbb{C} \setminus \{iy|y \le 0\}$ by

$$\log z = \log|z| + i\arg z,$$

where $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$. Then

$$\int_C \frac{\log z}{1+z^2}dz = 2\pi i \mathrm{Res}_{z=i}\frac{\log z}{1+z^2}$$
$$= 2\pi i \frac{i\frac{\pi}{2}}{2i}$$
$$= \frac{\pi^2}{2}i,$$

where C is the same curve as in **Fig.4-13** on Page 160. On the other hand, let γ be the upper half semicircle with radius R, then

$$\left|\int_\gamma \frac{\log z}{1+z^2}dz\right| \le \int_\gamma \frac{|\log z|}{|1+z^2|}|dz|$$
$$\le \pi R\frac{|\log|R|| \ + \ \pi}{|1-R^2|},$$

which tends to 0 in both cases $R \to 0$ and $R \to \infty$. Thus

$$\int_C \frac{\log z}{1+z^2}dz = \int_{-\infty}^0 \frac{\log|x| + i\pi}{1+x^2}dx + \int_0^\infty \frac{\log x}{1+x^2}dx$$
$$= I_1 + I_2.$$

Take real part in both sides, we get

$$0 = \mathrm{Re}(I_1) + I_2.$$

Note that

$$\mathrm{Re}(I_1) = \int_{-\infty}^{0} \frac{\log|x|}{1+x^2}dx = I_2,$$

we get

$$\int_0^\infty \frac{\log x}{1+x^2}dx = I_2 = 0.$$

N.B. If we are not restricted to use residue to compute this integral, we can get the result without any difficulty by changing variable

$$x \longrightarrow t = \frac{1}{x}.$$