

THE SINE PRODUCT FORMULA AND THE GAMMA FUNCTION

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ABSTRACT. The function $\sin x$ is very important in mathematics and has many applications. In addition to its series expansion, it can also be written as an infinite product. The infinite product of $\sin x$ can be used to prove certain values of $\zeta(s)$, such as $\zeta(2)$ and $\zeta(4)$. The gamma function is related directly to the $\sin x$ function and can be used to prove the infinite product expansion. Also used are Weierstrass' product formula and Legendre's relation.

1. INTRODUCTION

There are a few special functions in mathematics that have particular significance and many applications. The *gamma function* is one of those functions. The gamma function can be defined as

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

We can also get the formula

$$(1) \quad \Gamma(x+1) = x\Gamma(x)$$

by replacing x with $x+1$ and integrating by parts.

In addition, since $\Gamma(1) = 1$, using Equation (1), by induction, we can relate the gamma function to the factorial formula

$$(2) \quad \Gamma(n) = (n-1)!.$$

The gamma function has the properties that it is log convex and monotonic, which will be used in a later proof.

Another important function in mathematics is the sine function. The trigonometric function $\sin x$ can be written as an infinite series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

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The function $\sin x$ can also be written as an infinite product expansion. The gamma function is directly related to the sine function. To derive the infinite product expansion of the sine function, the Weierstrass product formula, Legendre relation, and the gamma function are all used.

The sine product formula is important in mathematics because it has many applications, including the proofs of other problems. One such application is the calculation of the values of $\zeta(2)$ and $\zeta(4)$, where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

is the *Riemann zeta function*.

Section 2 derives Weierstrass' product formula and Euler's constant. Section 3 introduces Stirling's formulas, Gauss' multiplication formula, and the Legendre relation. In Section 4, the sine product formula is produced from the gamma function. Finally, Section 5 discuss the applications of the sine product formula, including the calculation of $\zeta(2)$ and $\zeta(4)$.

2. WEIERSTRASS' PRODUCT FORMULA

Weierstrass derived a formula which, when applied to the gamma function, can be used to prove the sine product formula. To find Weierstrass' product formula, we first begin with a theorem.

Theorem 2.1. *The function $\Gamma(x)$ is equal to the limit as n goes to infinity of*

$$(3) \quad \Gamma(x) = \lim_{n \rightarrow \infty} \frac{n^x n!}{x(x+1) \cdots (x+n)}.$$

Proof. Begin with a difference quotient expressed as the inequality

$$\frac{\log \Gamma(-1+n) - \log \Gamma(n)}{(-1+n) - n} \leq \frac{\log \Gamma(x+n) - \log \Gamma(n)}{(x+n) - n} \leq \frac{\log \Gamma(1+n) - \log \Gamma(n)}{(1+n) - n}.$$

This inequality is true because of the gamma functions properties that it is monotonically increasing and log convex. Substitute for $\Gamma(n)$ using Equation (2) and simplify to get

$$\log(n-1) \leq \frac{\log \Gamma(x+n) - \log(n-1)!}{x} \leq \log n.$$

Rearrange the terms of the inequality and since the logarithm is a monotonically increasing function, we get

$$(n-1)^x(n-1)! \leq \Gamma(x+n) \leq n^x(n-1)!.$$

Using Equation (1), we know that $\Gamma(x+n) = (x+n-1)(x+n-2)\cdots(x+1)x\Gamma(x)$. Substituting it into the inequality gives us

$$\frac{(n-1)^x(n-1)!}{x(x+1)\cdots(x+n-1)} \leq \Gamma(x) \leq \frac{n^x n!}{x(x+1)\cdots(x+n)} \frac{x+n}{n}.$$

If we replace n with $n+1$ on the left hand side and then rearrange the inequality, we get

$$\Gamma(x) \frac{n}{x+n} \leq \frac{n^x n!}{x(x+1)\cdots(x+n)} \leq \Gamma(x).$$

Taking the limit as $n \rightarrow \infty$ gives us

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n^x n!}{x(x+1)\cdots(x+n)}.$$

□

Equation (3) was derived by Gauss. From there, Weierstrass was able to derive another form of the same equation

$$\Gamma(x) = e^{x(\log n - 1/2 - \dots - 1/n)} \frac{1}{x} \frac{e^{x/1}}{1+x/1} \cdot \frac{e^{x/2}}{1+x/2} \cdots \frac{e^{x/n}}{1+x/n}.$$

The limit of

$$\lim_{n \rightarrow \infty} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right)$$

exists, is equal to C , and is often called *Euler's constant*. So we can rewrite Weierstrass' product formula as

$$(4) \quad \Gamma(x) = e^{-Cx} \frac{1}{x} \prod_{i=1}^{\infty} \frac{e^{x/i}}{1+x/i}.$$

3. MULTIPLICATION FORMULA

There are three formulas,

$\Gamma(x) = \sqrt{2\pi} x^{x-1/2} e^{-x+\mu(x)}$, where

$$\mu(x) = \sum_{n=0}^{\infty} \left(x+n+\frac{1}{2} \right) \log \left(1 + \frac{1}{x+n} \right) - 1 = \frac{\theta}{12x}, \quad \text{and}$$

$$n! = \sqrt{2\pi n} n^{n+1/2} e^{-n+\theta/12n},$$

which are known as Stirling's formulas. Where θ is a number independent of other values and where $0 \leq \theta \leq 1$. Stirling's formulas are approximations of $\Gamma(x)$ for large values of x and the accuracy increases as x increases.

Gauss discovered a formula, which expresses $\Gamma(x)$ as a product of its factors. Gauss' multiplication formula is

$$(5) \quad \frac{(2\pi)^{(p-1)/2}}{p^{x-1/2}} \Gamma(x) = \Gamma\left(\frac{x}{p}\right) \Gamma\left(\frac{x+1}{p}\right) \cdots \Gamma\left(\frac{x+p-1}{p}\right),$$

where p is a positive integer.

There is the special case discovered by Legendre, where $p = 2$, which is called Legendre's relation. Legendre's relation states

$$(6) \quad \Gamma\left(\frac{x}{2}\right) \Gamma\left(\frac{x+1}{2}\right) = \frac{\sqrt{\pi}}{2^{x-1}} \Gamma(x).$$

The derivation and proof of these formulas can be found at [1]. They are based on finding an approximation for $\Gamma(x)$ in terms of an estimate for $n!$.

4. THE SINE AND AND GAMMA FUNCTIONS

To derive the sine product formula, we first find a relationship between the sine and gamma functions. We define a function $\phi(x)$ and find that $\phi(x+1) = \phi(x)$.

Theorem 4.1. *Define the function $\phi(x)$, for nonintegral x , to be*

$$(7) \quad \phi(x) = \Gamma(x) \Gamma(1-x) \sin \pi x,$$

then $\phi(x+1) = \phi(x)$.

Proof. If we use Equation (1) and substitute $-x+1$ for x , then we get that

$$(8) \quad \Gamma(-x+1) = -x \Gamma(-x).$$

Finding $\phi(x+1)$, we get

$$\phi(x+1) = \Gamma(x+1) \Gamma(-x) \sin(\pi(x+1)).$$

Note that $\sin(\pi x + \pi) = -\sin \pi x$. Use Equation (1), rearrange Equation (8), and substitute them in to get

$$\phi(x+1) = x \Gamma(x) \frac{\Gamma(-x+1)}{-x} (-\sin \pi x) = \Gamma(x) \Gamma(-x+1) \sin \pi x.$$

This is equal to the original equation for $\phi(x)$, so $\phi(x+1) = \phi(x)$. \square

The Legendre relation (Equation (6)) can be written as

$$\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = b2^{-x}\Gamma(x),$$

where $b = 2\sqrt{\pi}$ is a constant. Replacing x with $1 - x$ gives

$$\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(1 - \frac{x}{2}\right) = b2^{x-1}\Gamma(1-x).$$

From there, we get that

$$\phi\left(\frac{x}{2}\right)\phi\left(\frac{x+1}{2}\right) = \Gamma\left(\frac{x}{2}\right)\Gamma\left(1 - \frac{x}{2}\right) \sin \frac{\pi x}{2} \Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{1-x}{2}\right) \cos \frac{\pi x}{2}.$$

Simplifying the above, we find

$$\phi\left(\frac{x}{2}\right)\phi\left(\frac{x+1}{2}\right) = \frac{b^2}{4}\Gamma(x)\Gamma(1-x) \sin \pi x = \frac{b^2}{4}\phi(x) = c\phi(x)$$

where $c = \frac{b^2}{4}$ is a constant.

Using Equation (1) and the infinite series expansion of $\sin x$, we get that

$$\begin{aligned} \phi(x) &= \frac{\Gamma(1+x)}{x}\Gamma(1-x) \left(\pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} + \dots \right) \\ &= \Gamma(1+x)\Gamma(1-x) \left(\pi - \frac{\pi^3 x^2}{3!} + \frac{\pi^5 x^4}{5!} - \frac{\pi^7 x^6}{7!} + \dots \right). \end{aligned}$$

The right hand side of the equation equals π when $x = 0$. From there we see that $\phi(0) = \pi$. Let $g(x)$ be a periodic function that is equal the second derivative of $\log \phi(x)$. It is periodic because $\log \phi(x) = \log(\Gamma(x)\Gamma(1-x) \sin \pi x)$ is periodic and so the second derivative will also be periodic. Since $g(x)$ is periodic, then it satisfies the equation

$$(9) \quad g(x) = \frac{1}{4}\left(g\left(\frac{x}{2}\right) + g\left(\frac{x+1}{2}\right)\right).$$

Since $g(x)$ is continuous on the interval $0 \leq x \leq 1$, it is bounded by a constant M , $|g(x)| \leq M$. Because $g(x)$ is periodic, it is bounded by M for all x . From Equation (9), we get that

$$|g(x)| \leq \frac{1}{4} \left(\left|g\left(\frac{x}{2}\right)\right| + \left|g\left(\frac{x+1}{2}\right)\right| \right) \leq \frac{M}{2}.$$

From this we see that $g(x)$ can actually be bounded by $\frac{M}{2}$. We can continue to repeat this process until the bound of $g(x)$ goes to 0. Therefore $g(x) = 0$, which means that $\log \phi(x)$ is a linear function, because $g(x) = 0$ is its second derivative. Since $\log \phi(x)$ is periodic, this implies

that it is a constant, which also implies that $\phi(x)$ is constant. We know that $\phi(0) = \pi$ and therefore $\phi(x)$ must equal π for all x .

Rearranging Equation (7) and using the fact that $\phi(x) = \pi$,

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Using (1), the above equation can be rewritten as

$$\sin \pi x = \frac{\pi}{-x\Gamma(x)\Gamma(-x)}.$$

The Weierstrass product formula allows us to replace the gamma function and rewrite $\sin \pi x$ as an infinite product expansion

$$(10) \quad \sin \pi x = \pi x \prod_{i=1}^{\infty} \left(1 - \frac{x^2}{i^2}\right).$$

5. APPLICATIONS OF THE SINE PRODUCT FORMULA

Applications of the sine product formula include the calculation of certain values of the Riemann zeta function. The proof that $\zeta(2) = \pi^2/6$ is often called Euler's Theorem.

Theorem 5.1. *The sum of the reciprocal of the perfect squares is $\pi^2/6$:*

$$\sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof. Consider the function $\sin x = 0$, which has an infinite number of roots $\pm\pi, \pm2\pi, \pm3\pi, \dots$. Using the infinite series expansion of $\sin x$ and dividing $\sin x$ by x gives us the infinite series

$$(11) \quad 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots = 0.$$

Applying Equation (10) and dividing by πx , we get the infinite product expansion

$$(12) \quad \frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\dots$$

When Equation (12) is expanded, the coefficient of x^2 will be

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \dots$$

Using Equation (11), we get

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \dots = \frac{1}{3!}.$$

Multiplying both sides of the equation by π^2 gives us

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$$

□

Using a similar method, we can also calculate $\zeta(4)$, which is the sum of the reciprocals of numbers to the fourth power.

Theorem 5.2. *The value of $\zeta(4)$ is $\pi^4/90$:*

$$\sum_1^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Proof. From the calculation of $\zeta(2)$, we know that

$$(13) \quad \sum_{i=1}^{\infty} \frac{1}{i^2\pi^2} = \frac{1}{6}.$$

If we square Equation (13), we get that

$$(14) \quad \left(\sum_{i=1}^{\infty} \frac{1}{i^2\pi^2} \right)^2 = \frac{1}{6^2}.$$

Expanding the left hand side of Equation (14) we get

$$(15) \quad \left(\sum_{i=1}^{\infty} \frac{1}{i^2\pi^2} \right)^2 = \sum \frac{1}{i^4\pi^4} + 2 \sum_{i<j} \frac{1}{i^2j^2\pi^4}.$$

If we use the sine infinite product expansion, Equation (12), we get that the coefficient of x^4 for $\frac{\sin x}{x}$ is equal to the sum of the product of $\frac{x^2}{i^2\pi^2}$. In other words, the coefficient of x^4 is

$$\sum_{i<j} \frac{1}{i^2j^2\pi^4}.$$

From Equation (11), we also know that the coefficient of x^4 must be $\frac{1}{120}$, so

$$(16) \quad \sum_{i < j} \frac{1}{i^2 j^2 \pi^4} = \frac{1}{120}.$$

Therefore, substituting Equation (16) into Equation (15) gives us

$$\left(\sum_{i=0}^{\infty} \frac{1}{i^2 \pi^2} \right)^2 = \frac{1}{36} = \sum \frac{1}{i^4 \pi^4} + 2 \left(\frac{1}{120} \right).$$

To calculate $\zeta(4)$, simply solve for $\sum \frac{1}{i^4 \pi^4}$ and multiply both sides by π^4 , to get

$$\sum_{i=1}^{\infty} \frac{1}{i^4} = \frac{\pi^4}{90}.$$

□

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