18.102 Introduction to Functional Analysis
Spring 2009

Lecture 14. Tuesday, March 31: Fourier series and $L^2(0,2\pi)$.

Fourier series. Let us now try applying our knowledge of Hilbert space to a concrete Hilbert space such as $L^2(a,b)$ for a finite interval $(a,b) \subset \mathbb{R}$. You showed that this is indeed a Hilbert space. One of the reasons for developing Hilbert space techniques originally was precisely the following result.

Theorem 12. *If $u \in L^2(0,2\pi)$ then the Fourier series of u,*

$$\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} c_k e^{ikx},\ c_k = \int_{(0,2\pi)} u(x)e^{-ikx}dx \tag{14.1}$$

converges in $L^2(0,2\pi)$ to u.

Notice that this does not say the series converges pointwise, or pointwise almost everywhere since this need not be true – depending on u. We are just claiming that

$$\lim_{n\to\infty}\int |u(x) - \frac{1}{2\pi}\sum_{|k|\le n} c_k e^{ikx}|^2 = 0 \tag{14.2}$$

for any $u \in L^2(0,2\pi)$.

First let's see that our abstract Hilbert space theory has put us quite close to proving this. First observe that if $e'_k(x) = \exp(ikx)$ then these elements of $L^2(0,2\pi)$ satisfy

$$\int e'_k\overline{e'_j} = \int_0^{2\pi}\exp(i(k-j)x) = \begin{cases} 0 & \text{if } k \ne j \\ 2\pi & \text{if } k = j. \end{cases} \tag{14.3}$$

Thus the functions

$$e_k = \frac{e'_k}{\|e'_k\|} = \frac{1}{\sqrt{2\pi}}e^{ikx} \tag{14.4}$$

form an orthonormal set in $L^2(0,2\pi)$. It follows that (14.1) is just the Fourier-Bessel series for u with respect to this orthonormal set:-

$$c_k = \sqrt{2\pi}\langle u, e_k\rangle \Longrightarrow \frac{1}{2\pi}c_k e^{ikx} = \langle u, e_k\rangle e_k. \tag{14.5}$$

So, we alreay know that this series converges in $L^2(0,2\pi)$ thanks to Bessel's identity. So 'all' we need to show is

Proposition 21. *The e_k, $k \in \mathbb{Z}$, form an orthonormal basis of $L^2(0,2\pi)$, i.e. are complete:*

$$\int ue^{ikx} = 0\ \forall\ k \Longrightarrow u = 0 \text{ in } L^2(0,2\pi). \tag{14.6}$$

This however, is not so trivial to prove. An equivalent statement is that the finite linear span of the e_k is *dense* in $L^2(0,2\pi)$. I will prove this using Fejér's method. In this approach, we check that any continuous function on $[0,2\pi]$ satisfying the additional condition that $u(0) = u(2\pi)$ is the uniform limit on $[0,2\pi]$ of a sequence in the finite span of the e_k. Since uniform convergence of continuous functions certainly implies convergence in $L^2(0,2\pi)$ and we already know that the continuous functions which vanish near 0 and 2π are dense in $L^2(0,2\pi)$ (I will recall why later) this is enough to prove Proposition 21. However the proof is a serious piece of analysis, at least it is to me!

So, the problem is to find the sequence in the span of the e_k. Of course the trick is to use the Fourier expansion that we want to check. The idea of Cesàro is to make this Fourier expansion 'converge faster', or maybe better. For the moment we can work with a general function $u \in L^2(0,2\pi)$ – or think of it as continuous if you prefer. So the truncated Fourier series is

$$U_n(x) = \frac{1}{2\pi}\sum_{|k|\le n}(\int_{(0,2\pi)} u(t)e^{-ikt}dt)e^{ikx} \tag{14.7}$$

where I have just inserted the definition of the c_k's into the sum. This is just a finite sum so we can treat x as a parameter and use the linearity of the integral to write this as

$$U_n(x) = \int_{(0,2\pi)} D_n(x-t)u(t),\ D_n(s) = \frac{1}{2\pi}\sum_{|k|\le n} e^{iks}. \tag{14.8}$$

Now this sum can be written as an explicit quotient, since, by telescoping,

$$(2\pi)D_n(s)(e^{is/2} - e^{is/2}) = e^{i(n+\frac{1}{2})s} - e^{-i(n+\frac{1}{2})s}. \tag{14.9}$$

So in fact, at least where $s \neq 0$,

$$D_n(s) = \frac{e^{i(n+\frac{1}{2})s} - e^{-i(n+\frac{1}{2})s}}{2\pi(e^{is/2} - e^{-is/2})} \tag{14.10}$$

and of course the limit as $s \to 0$ exists just fine.

As I said, Cesàro's idea is to speed up the convergence by replacing U_n by its average

$$V_n(x) = \frac{1}{n+1}\sum_{l=0}^{n} U_l. \tag{14.11}$$

Again plugging in the definitions of the U_l's and using the linearity of the integral we see that

$$V_n(x) = \int_{(0,2\pi)} S_n(x-t)u(t),\ S_n(s) = \frac{1}{n+1}\sum_{l=0}^{n} D_l(s). \tag{14.12}$$

So again we want to compute a more useful form for $S_n(s)$ – which is the Fejér kernel. Since the denominators in (14.10) are all the same,

$$2\pi(n+1)(e^{is/2} - e^{-is/2})S_n(s) = \sum_{l=0}^{n} e^{i(n+\frac{1}{2})s} - \sum_{l=0}^{n} e^{-i(n+\frac{1}{2})s}. \tag{14.13}$$

Using the same trick again,

$$(e^{is/2} - e^{-is/2})\sum_{l=0}^{n} e^{i(n+\frac{1}{2})s} = e^{i(n+1)s} - 1 \tag{14.14}$$

so

$$2\pi(n+1)(e^{is/2} - e^{-is/2})^2 S_n(s) = e^{i(n+1)s} + e^{-i(n+1)s} - 2 \Longrightarrow \\ S_n(s) = \frac{1}{n+1}\frac{\sin^2(\frac{(n+1)}{2}s)}{2\pi\sin^2(\frac{s}{2})}. \tag{14.15}$$

Now, what can we say about this function? One thing we know immediately is that if we plug $u = 1$ into the disucssion above, we get $U_n = 1$ for $n \geq 0$ and hence $V_n = 1$ as well. Thus in fact

$$\int_{(0,2\pi)} S_n(x - \cdot) = 1. \tag{14.16}$$

Now looking directly at (14.15) the first thing to notice is that $S_n(s) \geq 0$. Also, we can see that the denominator only vanishes when $s = 0$ or $s = 2\pi$ in $[0, 2\pi]$. Thus if we stay away from there, say $s \in (\delta, 2\pi - \delta)$ for some $\delta > 0$ then – since sin is a bounded function

$$|S_n(s)| \leq (n + 1)^{-1} C_\delta \text{ on } (\delta, 2\pi - \delta). \tag{14.17}$$

Now, we are interested in how close $V_n(x)$ is to the given $u(x)$ in supremum norm, where now we will take u to be continuous. Because of (14.16) we can write

$$u(x) = \int_{(0,2\pi)} S_n(x - t)u(x) \tag{14.18}$$

where t denotes the variable of integration (and x is fixed in $[0, 2\pi]$). This 'trick' means that the difference is

$$V_n(x) - u(x) = \int_{(0,2\pi)} S_x(x - t)(u(t) - u(x)). \tag{14.19}$$

For each x we split this integral into two parts, the set $\Gamma(x)$ where $x - t \in [0, \delta]$ or $x - t \in [2\pi - \delta, 2\pi]$ and the remainder. So

$$|V_n(x) - u(x)| \leq \int_{\Gamma(x)} S_x(x - t)|u(t) - u(x)| + \int_{(0,2\pi)\setminus\Gamma(x)} S_x(x - t)|u(t) - u(x)|. \tag{14.20}$$

Now on $\Gamma(x)$ either $|t - x| \leq \delta$ – the points are close together – or t is close to 0 and x to 2π so $2\pi - x + t \leq \delta$ or conversely, x is close to 0 and t to 2π so $2\pi - t + x \leq \delta$. In any case, by assuming that $u(0) = u(2\pi)$ and using the uniform continuity of a continuous function on $[0, 2\pi]$, given $\epsilon > 0$ we can choose δ so small that

$$|u(x) - u(t)| \leq \epsilon/2 \text{ on } \Gamma(x). \tag{14.21}$$

On the complement of $\Gamma(x)$ we have (14.17) and since u is bounded we get the estimate

$$|V_n(x) - u(x)| \leq \epsilon/2 \int_{\Gamma(x)} S_n(x - t) + (n + 1)^{-1} C'(\delta) \leq \epsilon/2 + (n + 1)^{-1} C'(\delta). \tag{14.22}$$

Here the fact that S_n is non-negative and has integral one has been used again to estimate the integral of $S_n(x - t)$ over $\Gamma(x)$ by 1. Thus, having chosen δ to make the first term small, we can choose n large to make the second term small and it follows that

$$V_n(x) \to u(x) \text{ uniformly on } [0, 2\pi] \text{ as } n \to \infty \tag{14.23}$$

under the assumption that $u \in \mathcal{C}([0, 2\pi])$ satisfies $u(0) = u(2\pi)$.

So this proves Proposition 21 subject to the density in $L^2(0, 2\pi)$ of the continuous functions which vanish near (but not of course in a fixed neighbourhood) of the ends.

In fact we know that the L^2 functions which vanish near the ends are dense since we can chop of and use the fact that

$$\lim_{\delta\to 0}\left(\int_{(0,\delta)}|f|^2+\int_{(2\pi-\delta,2\pi)}|f|^2\right)=0. \tag{14.24}$$

The L^2 functions which vanish near the ends are in the closure of the span of the step functions which vanish near the ends. Each such step function can be approximated in $L^2((0,2\pi))$ by a continuous function which vanishes near the ends so we are done as far as density is concerned. So we have proved Theorem 12.

Problem set 7, Due 11AM Tuesday 7 Apr.

I will put up some practice problems for the test next Thursday when I get a chance.

Problem 7.1 Question:- Is it possible to show the completeness of the Fourier basis

$$\exp(ikx)/\sqrt{2\pi}$$

by computation? Maybe, see what you think. These questions are also intended to get you to say things clearly.

(1) Work out the Fourier coefficients $c_k(t)=\int_{(0,2\pi)} f_t e^{-ikx}$ of the step function

$$f_t(x)=\begin{cases}1 & 0\le x<t\\ 0 & t\le x\le 2\pi\end{cases} \tag{14.25}$$

for each fixed $t\in(0,2\pi)$.

(2) Explain why this Fourier series converges to f_t in $L^2(0,2\pi)$ if and only if

$$2\sum_{k>0}|c_k(t)|^2=2\pi t-t^2,\ t\in(0,2\pi). \tag{14.26}$$

(3) Write this condition out as a Fourier series and apply the argument again to show that the completeness of the Fourier basis implies identities for the sum of k^{-2} and k^{-4}.

(4) Can you explain how reversing the argument, that knowledge of the sums of these two series should imply the completeness of the Fourier basis? There is a serious subtlety in this argument, and you get full marks for spotting it, without going ahead a using it to prove completeness.

Problem 7.2 Prove that for appropriate constants d_k, the functions $d_k\sin(kx/2)$, $k\in\mathbb{N}$, form an orthonormal basis for $L^2(0,2\pi)$.

Hint: The usual method is to use the basic result from class plus translation and rescaling to show that $d'_k\exp(ikx/2)$ $k\in\mathbb{Z}$ form an orthonormal basis of $L^2(-2\pi,2\pi)$. Then extend functions as odd from $(0,2\pi)$ to $(-2\pi,2\pi)$.

Problem 7.3 Let e_k, $k\in\mathbb{N}$, be an orthonormal basis in a separable Hilbert space, H. Show that there is a uniquely defined bounded linear operator $S:H\longrightarrow H$, satisfying

$$Se_j=e_{j+1}\ \forall\ j\in\mathbb{N}. \tag{14.27}$$

Show that if $B:H\longrightarrow H$ is a bounded linear operator then $S+\epsilon B$ is *not* invertible if $\epsilon<\epsilon_0$ for some $\epsilon_0>0$.

Hint:- Consider the linear functional $L:H\longrightarrow\mathbb{C}$, $Lu=(Bu,e_1)$. Show that $B'u=Bu-(Lu)e_1$ is a bounded linear operator from H to the Hilbert space

$H_1 = \{u \in H; (u, e_1) = 0\}$. Conclude that $S + \epsilon B'$ is invertible as a linear map from H to H_1 for small ϵ. Use this to argue that $S + \epsilon B$ cannot be an isomorphism from H to H by showing that either e_1 is not in the range or else there is a non-trivial element in the null space.

Problem 7.4 Show that the product of bounded operators on a Hilbert space is strong continuous, in the sense that if A_n and B_n are strong convergent sequences of bounded operators on H with limits A and B then the product $A_n B_n$ is strongly convergent with limit AB.

Hint: Be careful! Use the result in class which was deduced from the Uniform Boundedness Theorem.

Solutions to Problems 6

Hint: Don't pay too much attention to my hints, sometimes they are a little off-the-cuff and may not be very helpfult. An example being the old hint for Problem 6.2!

Problem 6.1 Let H be a separable Hilbert space. Show that $K \subset H$ is compact if and only if it is closed, bounded and has the property that any sequence in K which is weakly convergent sequence in H is (strongly) convergent.

Hint:- In one direction use the result from class that any bounded sequence has a weakly convergent subsequence.

Problem 6.2 Show that, in a separable Hilbert space, a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if the weak limit, v satisfies

$$\|v\|_H = \lim_{n\to\infty} \|v_n\|_H. \tag{14.28}$$

Hint:- To show that this condition is sufficient, expand

$$(v_n - v, v_n - v) = \|v_n\|^2 - 2\operatorname{Re}(v_n, v) + \|v\|^2. \tag{14.29}$$

Problem 6.3 Show that a subset of a separable Hilbert space is compact if and only if it is closed and bounded and has the property of 'finite dimensional approximation' meaning that for any $\epsilon > 0$ there exists a linear subspace $D_N \subset H$ of finite dimension such that

$$d(K, D_N) = \sup_{u\in K} \inf_{v\in D_N} \{d(u, v)\} \le \epsilon. \tag{14.30}$$

Hint:- To prove necessity of this condition use the 'equi-small tails' property of compact sets with respect to an orthonormal basis. To use the finite dimensional approximation condition to show that any weakly convergent sequence in K is strongly convergent, use the convexity result from class to define the sequence $\{v'_n\}$ in D_N where v'_n is the closest point in D_N to v_n. Show that v'_n is weakly, hence strongly, convergent and hence deduce that $\{v_n\}$ is Cauchy.

Problem 6.4 Suppose that $A : H \longrightarrow H$ is a bounded linear operator with the property that $A(H) \subset H$ is finite dimensional. Show that if v_n is weakly convergent in H then Av_n is strongly convergent in H.

Problem 6.5 Suppose that H_1 and H_2 are two different Hilbert spaces and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint) $A^* : H_2 \longrightarrow H_1$ with the property

$$\langle Au_1, u_2\rangle_{H_2} = \langle u_1, A^*u_2\rangle_{H_1} \ \forall\ u_1 \in H_1,\ u_2 \in H_2. \tag{14.31}$$