18.102 Introduction to Functional Analysis
Spring 2009

Apendix. Exam Preparation Problems

EP.1 Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and suppose that

$$B : H \times H \longleftrightarrow \mathbb{C} \tag{1}$$

is a(nother) sesquilinear form – so for all $c_1,\ c_2 \in \mathbb{C}$, u, u_1, u_2 and $v \in H$,

$$B(c_1u_1 + c_2u_2, v) = c_1B(u_1, v) + c_2B(u_2, v),\ B(u, v) = \overline{B(v, u)}. \tag{2}$$

Show that B is continuous, with respect to the norm $\|(u, v)\| = \|u\|_H + \|v\|_H$ on $H \times H$ if and only if it is bounded, in the sense that for some $C > 0$,

$$|B(u, v)| \le C\|u\|_H\|v\|_H. \tag{3}$$

EP.2 A continuous linear map $T : H_1 \longrightarrow H_2$ between two, possibly different, Hilbert spaces is said to be *compact* if the image of the unit ball in H_1 under T is precompact in H_2. Suppose $A : H_1 \longrightarrow H_2$ is a continuous linear operator which is injective and surjective and $T : H_1 \longrightarrow H_2$ is compact. Show that there is a compact operator $K : H_2 \longrightarrow H_2$ such that $T = KA$.

EP.3 Suppose $P \subset H$ is a (non-trivial, i.e. not $\{0\}$) closed linear subspace of a Hilbert space. Deduce from a result done in class that each u in H has a unique decomposition

$$u = v + v',\ v \in P,\ v' \perp P \tag{4}$$

and that the map $\pi_P : H \ni u \longmapsto v \in P$ has the properties

$$(\pi_P)^* = \pi_P,\ (\pi_P)^2 = \pi_P,\ \|\pi_P\|_{\mathcal{B}(H)} = 1. \tag{5}$$

EP.4 Show that for a sequence of non-negative step functions f_j, defined on $\mathbb{R}$, which is absolutely summable, meaning $\sum\limits_j \int f_j < \infty$, the series $\sum\limits_j f_j(x)$ cannot diverge for all $x \in (a, b)$, for any $a < b$.

EP.5 Let $A_j \subset [-N, N] \subset \mathbb{R}$ (for N fixed) be a sequence of subsets with the property that the characteristic function, χ_j of A_j, is integrable for each j. Show that the characteristic function of

$$A = \bigcup_j A_j \tag{6}$$

is integrable.

EP.6 Let $e_j = c_jC^je^{-x^2/2}$, $c_j > 0$, $C = -\frac{d}{dx} + x$ the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. Define an operator on $L^2(\mathbb{R})$ by

$$Au = \sum_{j=0}^{\infty}(2j + 1)^{-\frac{1}{2}}(u, e_j)_{L^2}e_j. \tag{7}$$

(1) Show that A is compact as an operator on $L^2(\mathbb{R})$.
(2) Suppose that $V \in \mathcal{C}^0_\infty(\mathbb{R})$ is a bounded, real-valued, continuous function on $\mathbb{R}$. What can you say about the eigenvalues τ_j, and eigenfunctions v_j, of $K = AVA$, where V is acting by multiplication on $L^2(\mathbb{R})$?
(3) What would you need to show to conclude that these eigenfunctions satisfy

$$-\frac{d^2v_j(x)}{dx^2} + x^2v_j(x) + V(x)v_j(x) = \lambda_jv_j? \tag{8}$$

(4) What would you need to show to check that all the square-integrable, twice continuously differentiable, solutions of (8), for some $\lambda_j \in \mathbb{C}$, are eigenfunctions of K?

EP.7 Test 1 from last year:-

Q1. Recall Lebesgue's Dominated Convergence Theorem and use it to show that if $u \in L^2(\mathbb{R})$ and $v \in L^1(\mathbb{R})$ then

$$\begin{gathered}\lim_{N\to\infty}\int_{|x|>N}|u|^2 = 0,\ \lim_{N\to\infty}\int|C_N u - u|^2 = 0,\\ \lim_{N\to\infty}\int_{|x|>N}|v| = 0 \text{ and } \lim_{N\to\infty}\int|C_N v - v| = 0.\end{gathered} \tag{Eq1}$$

where

$$C_N f(x) = \begin{cases} N & \text{if } f(x) > N \\ -N & \text{if } f(x) < -N \\ f(x) & \text{otherwise.}\end{cases} \tag{Eq2}$$

Q2. Show that step functions are dense in $L^1(\mathbb{R})$ and in $L^2(\mathbb{R})$ (Hint:- Look at Q1 above and think about $f - f_N$, $f_N = C_N f\chi_{[-N,N]}$ and its square. So it suffices to show that f_N is the limit in L^2 of a sequence of step functions. Show that if g_n is a sequence of step functions converging to f_N in L^1 then $C_N\chi_{[-N,N]}$ is converges to f_N in L^2.) and that if $f \in L^1(\mathbb{R})$ then there is a sequence of step functions u_n and an element $g \in L^1(\mathbb{R})$ such that $u_n \to f$ a.e. and $|u_n| \le g$.

Q3. Show that $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ are separable, meaning that each has a countable dense subset.

Q4. Show that the minimum and the maximum of two locally integrable functions is locally integrable.

Q5. A subset of $\mathbb{R}$ is said to be (Lebesgue) measurable if its characteristic function is locally integrable. Show that a countable union of measurable sets is measurable. Hint: Start with two!

Q6. Define $L^\infty(\mathbb{R})$ as consisting of the locally integrable functions which are bounded, $\sup_{\mathbb{R}}|u| < \infty$. If $\mathcal{N}_\infty \subset L^\infty(\mathbb{R})$ consists of the bounded functions which vanish outside a set of measure zero show that

$$\|u + \mathcal{N}_\infty\|_{L^\infty} = \inf_{h\in\mathcal{N}_\infty}\sup_{x\in\mathbb{R}}|u(x) + h(x)| \tag{Eq3}$$

is a norm on $\mathcal{L}^\infty(\mathbb{R}) = L^\infty(\mathbb{R})/\mathcal{N}_\infty$.

Q7. Show that if $u \in L^\infty(\mathbb{R})$ and $v \in L^1(\mathbb{R})$ then $uv \in L^1(\mathbb{R})$ and that

$$|\int uv| \le \|u\|_{L^\infty}\|v\|_{L^1}. \tag{Eq4}$$

Q8. Show that each $u \in L^2(\mathbb{R})$ is continuous in the mean in the sense that $T_z u(x) = u(x-z) \in L^2(\mathbb{R})$ for all $z \in \mathbb{R}$ and that

$$\lim_{|z|\to 0}\int|T_z u - u|^2 = 0. \tag{Eq5}$$

Q9. If $\{u_j\}$ is a Cauchy sequence in $L^2(\mathbb{R})$ show that both (Eq5) and (Eq1) are uniform in j, so given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int |T_z u_j - u_j|^2 < \epsilon, \ \int_{|x|>1/\delta} |u_j|^2 < \epsilon \ \forall \ |z| < \delta \text{ and all } j. \tag{Eq6}$$

Q10. Construct a sequence in $L^2(\mathbb{R})$ for which the uniformity in (Eq6) does not hold.

EP.8 Test 2 from last year.

(1) Recall the discussion of the Dirichlet problem for d^2/dx^2 from class and carry out an analogous discussion for the Neumann problem to arrive at a complete orthonormal basis of $L^2([0,1])$ consisting of $\psi_n \in \mathcal{C}^2$ functions which are all eigenfunctions in the sense that

$$\frac{d^2\psi_n(x)}{dx^2} = \gamma_n \psi_n(x) \ \forall \ x \in [0,1], \ \frac{d\psi_n}{dx}(0) = \frac{d\psi_n}{dx}(1) = 0. \tag{NeuEig}$$

This is actually a little harder than the Dirichlet problem which I did in class, because there is an eigenfunction of norm 1 with $\gamma = 0$. Here are some individual steps which may help you along the way!

What is the eigenfunction with eigenvalue 0 for (NeuEig)?

What is the operator of orthogonal projection onto this function?

What is the operator of orthogonal projection onto the orthocomplement of this function?

The crucual part. Find an integral operator $A_N = B - B_N$, where B is the operator from class,

$$(Bf)(x) = \int_0^x (x-s) f(s) ds \tag{B-Def}$$

and B_N is of finite rank, such that if f is continuous then $u = A_N f$ is twice continuously differentiable, satisfies $\int_0^1 u(x)dx = 0$, $A_N 1 = 0$ (where 1 is the constant function) and

$$\int_0^1 f(x)dx = 0 \Longrightarrow \\ \frac{d^2u}{dx^2} = f(x) \ \forall \ x \in [0,1], \ \frac{du}{dx}(0) = \frac{du}{dx}(1) = 0. \tag{GI}$$

Show that A_N is compact and self-adjoint.

Work out what the spectrum of A_N is, including its null space.

Deduce the desired conclusion.

(2) Show that these two orthonormal bases of $L^2([0,1])$ (the one above and the one from class) can each be turned into an orthonormal basis of $L^2([0,\pi])$ by change of variable.

(3) Construct an orthonormal basis of $L^2([-\pi,\pi])$ by dividing each element into its odd and even parts, resticting these to $[0,\pi]$ and using the Neumann basis above on the even part and the Dirichlet basis from class on the odd part.

(4) Prove the basic theorem of Fourier series, namely that for any function $u \in L^2([-\pi,\pi])$ there exist unique constants $c_k \in \mathbb{C}$, $k \in \mathbb{Z}$ such that

$$u(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx} \text{ converges in } L^2([-\pi,\pi]) \tag{FS}$$

and give an integral formula for the constants.