18.102 Introduction to Functional Analysis
Spring 2009

PRELIMINARY PROBLEMS FOR TEST 2 FOR 18.102, SPRING 2009
TEST ON THURSDAY 9 APR.

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1. Problem T2.1

Let H be a separable Hilbert space with an orthonormal basis e_i, $i \in \mathbb{N}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$:

(1) Give an example of a sequence u_n which is *not* weakly convergent in H but is such that (u_n, e_j) converges for each j.

(2) Show that if $\|u_n\|$ is bounded and (u_n, e_j) converges for each j then $u_n \rightharpoonup u$ converges weakly.

2. Problem T2.2

Let h_1 be the space of sequences

$$h_1 = \{c : \mathbb{N} \longrightarrow \mathbb{C}; \|c\|_{h_1}^2 = \sum_{j=1}^{\infty} j^2 |c_j|^2 < \infty\}. \tag{1}$$

(1) Show that h_1 is a Hilbert space.

(2) Show that the unit ball in h_1 is *pre-compact* in the standard Hilbert space l_2 – meaning its closure in l_2 is compact.

3. Problem T2.3

(1) Recall the definition of a subset of $\mathbb{R}$ of measure zero in terms of convergence of series of step functions.

(2) Show directly that a set of measure zero cannot contain a non-trivial open interval.

(3) Show that the complement of a set of measure zero is dense in $\mathbb{R}$.

4. Problem T2.4

Suppose that $f \in L^2(0, 2\pi)$ is such that its Fourier coefficients

$$c_k = \int_{(0,2\pi)} f(x) e^{-ikx} \tag{1}$$

satisfy

$$\sum_k (k+1)^2 \|c_k\|^2 < \infty. \tag{2}$$

Prove that there is a continuous function $\tilde{f} : [0, 2\pi] \longrightarrow \mathbb{C}$ with $\tilde{f}(0) = \tilde{f}(2\pi)$ such that $[f] = [\tilde{f}]$ in $L^2(0, 2\pi)$.

5. Problem T2.5

Let h_1 be the Hilbert space in Problem T2.2. Show that any linear functional satisfying

$$T : h_1 \longrightarrow \mathbb{C},\ |Tc| \le C\|c\|_{h_1} \tag{1}$$

for some constant C is of the form

$$Tc = \sum_{j=1}^{\infty} c_i d_i \tag{2}$$

for a fixed sequence d_i satisfying

$$\sum_{k=1}^{\infty} k^{-2}|d_k|^2 < \infty. \tag{3}$$

6. Problem T2.6

7. Problem T2.7

8. Problem T2.8

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