18.102 Introduction to Functional Analysis
Spring 2009

PROBLEM SET 10 (AND LAST) FOR 18.102, SPRING 2009 DUE 11AM TUESDAY 5 MAY.

RICHARD MELROSE

By now you should have become reasonably comfortable with a separable Hilbert space such as l_2. However, it is worthwhile checking once again that it is rather large – if you like, let me try to make you uncomfortable for one last time. An important result in this direction is Kuiper's theorem, which I will *not* ask you to prove[1]. However, I want you to go through the closely related result sometimes known as *Eilenberg's swindle.* Perhaps you will appreciate the little bit of trickery. First some preliminary results. Note that everything below is a closed curve in the $x \in [0, 1]$ variable – you might want to identify this with a circle instead, I just did it the primitive way.

Problem P10.1

Let H be a separable, infinite dimensional Hilbert space. Show that the direct sum of two copies of H is a Hilbert space with the norm

$$H \oplus H \ni (u_1, u_2) \longmapsto (\|u_1\|_H^2 + \|u_2\|_H^2)^{\frac{1}{2}} \tag{P10.1}$$

either by constructing an isometric isomorphism

$$T : H \longrightarrow H \oplus H, \text{ 1-1 and onto, } \|u\|_H = \|Tu\|_{H\oplus H} \tag{P10.2}$$

or otherwise. In any case, construct a map as in (P10.2).

Problem P10.2

One can repeat the preceding construction any finite number of times. Show that it can be done 'countably often' in the sense that if H is a separable, infinite dimensional, Hilbert space then

$$l_2(H) = \{u : \mathbb{N} \longrightarrow H; \|u\|_{l_2(H)}^2 = \sum_i \|u_i\|_H^2 < \infty\} \tag{P10.3}$$

has a Hilbert space structure and construct an explicit isometric isomorphism from $l_2(H)$ to H.

[1]Kuiper's theorem says that for any (norm) continuous map, say from any compact metric space, $g : M \longrightarrow \mathrm{GL}(H)$ with values in the invertible operators on a separable infinite-dimensional Hilbert space there exists a continuous map, an homotopy, $h : M \times [0, 1] \longrightarrow \mathrm{GL}(H)$ such that $h(m, 0) = g(m)$ and $h(m, 1) = \mathrm{Id}_H$ for all $m \in M$.

Problem P10.3

Recall, or perhaps learn about, the winding number of a closed curve with values in $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. We take as given the following fact:[2] If $Q = [0,1]^N$ and $f : Q \longrightarrow \mathbb{C}^*$ is continuous then for each choice of $b \in \mathbb{C}$ satisfying $\exp(2\pi ib) = f(0)$, there exists a unique continuous function $F : Q \longrightarrow \mathbb{C}$ satisfying

$$\exp(2\pi iF(q)) = f(q),\ \forall\ q \in Q \text{ and } F(0) = b. \tag{P10.4}$$

Of course, you are free to change b to $b+n$ for any $n \in \mathbb{Z}$ but then F changes to $F+n$, just shifting by the same integer.

(1) Now, suppose $c : [0,1] \longrightarrow \mathbb{C}^*$ is a closed curve – meaning it is continuous and $c(1) = c(0)$. Let $C : [0,1] \longrightarrow \mathbb{C}$ be a choice of F for $N = 1$ and $f = c$. Show that the winding number of the closed curve c may be defined *unambiguously* as

$$\mathrm{wn}(c) = F(1) - F(0) \in \mathbb{Z}. \tag{P10.5}$$

(2) Show that $\mathrm{wn}(c)$ is *constant under homotopy.* That is if $c_i : [0,1] \longrightarrow \mathbb{C}^*$, $i = 1,2$, are two closed curves so $c_i(1) = c_i(0)$, $i = 1,2$, which are *homotopic* through closed curves in the sense that there exists $f : [0,1]^2 \longrightarrow \mathbb{C}^*$ continuous and such that $f(0,x) = c_1(x)$, $f(1,x) = c_2(x)$ for all $x \in [0,1]$ and $f(y,0) = f(y,1)$ for all $y \in [0,1]$, then $\mathrm{wn}(c_1) = \mathrm{wn}(c_2)$.

(3) Consider the closed curve $L_n : [0,1] \ni x \longmapsto e^{2\pi ix}\,\mathrm{Id}_{n\times n}$ of $n\times n$ matrices. Using the standard properties of the determinant, show that this curve is not homotopic to the identity through closed curves in the sense that there does not exist a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(n)$, with values in the invertible $n\times n$ matrices, such that $G(0,x) = L_n(x)$, $G(1,x) \equiv \mathrm{Id}_{n\times n}$ for all $x \in [0,1]$, $G(y,0) = G(y,1)$ for all $y \in [0,1]$.

Problem P10.4

Consider the closed curve corresponding to L_n above in the case of a separable but now infinite dimensional Hilbert space:

$$L : [0,1] \ni x \longmapsto e^{2\pi ix}\,\mathrm{Id}_H \in \mathrm{GL}(H) \subset \mathcal{B}(H) \tag{P10.6}$$

taking values in the invertible operators on H. Show that after identifying H with $H \oplus H$ as above, there is a continuous map

$$M : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{P10.7}$$

with values in the invertible operators and satisfying

$$M(0,x) = L(x),\ M(1,x)(u_1,u_2) = (e^{4\pi ix}u_1, u_2),\ M(y,0) = M(y,1),\ \forall\ x,y \in [0,1]. \tag{P10.8}$$

Hint: So, think of $H \oplus H$ as being 2-vectors (u_1,u_2) with entries in H. This allows one to think of 'rotation' between the two factors. Indeed, show that

$$U(y)(u_1,u_2) = (\cos(\pi y/2)u_1 + \sin(\pi y/2)u_2, -\sin(\pi y/2)u_1 + \cos(\pi y/2)u_2) \tag{P10.9}$$

defines a continuous map $[0,1] \ni y \longmapsto U(y) \in \mathrm{GL}(H\oplus H)$ such that $U(0) = \mathrm{Id}$, $U(1)(u_1,u_2) = (u_2,-u_1)$. Now, consider the 2-parameter family of maps

$$U^{-1}(y)V_2(x)U(y)V_1(x) \tag{P10.10}$$

[2] Of course, you are free to give a proof – it is not hard.

where $V_1(x)$ and $V_2(x)$ are defined on $H \oplus H$ as multiplication by $\exp(2\pi ix)$ on the first and the second component respectively, leaving the other fixed.

Problem P10.5

Using a rotation similar to the one in the preceeding problem (or otherwise) show that there is a continuous map

$$G : [0,1]^2 \longrightarrow \mathrm{GL}(H \oplus H) \tag{P10.11}$$

such that

$$G(0,x)(u_1,u_2) = (e^{2\pi ix}u_1, e^{-2\pi ix}u_2),\\ G(1,x)(u_1,u_2) = (u_1,u_2),\ G(y,0) = G(y,1)\ \forall\ x,y \in [0,1]. \tag{P10.12}$$

Problem P10.6

Now, think about combining the various constructions above in the following way. Show that on $l_2(H)$ there is an homotopy like (P10.11), $\tilde{G} : [0,1]^2 \longrightarrow \mathrm{GL}(l_2(H))$, (very like in fact) such that

$$\tilde{G}(0,x)\left\{u_k\right\}_{k=1}^{\infty} = \left\{\exp((-1)^k 2\pi ix)u_k\right\}_{k=1}^{\infty},\\ \tilde{G}(1,x) = \mathrm{Id},\ \tilde{G}(y,0) = \tilde{G}(y,1)\ \forall\ x,y \in [0,1]. \tag{P10.13}$$

Problem P10.7: Eilenberg's swindle

For an infinite dimenisonal separable Hilbert space, construct an homotopy – meaning a continuous map $G : [0,1]^2 \longrightarrow \mathrm{GL}(H)$ – with $G(0,x) = L(x)$ in (P10.6) and $G(1,x) = \mathrm{Id}$ and of course $G(y,0) = G(y,1)$ for all $x,y \in [0,1]$.

Hint: Just put things together – of course you can rescale the interval at the end to make it all happen over $[0,1]$. First 'divide H into 2 copies of itself' and deform from L to $M(1,x)$ in (P10.8). Now, 'divide the second H up into $l_2(H)$' and apply an argument just like the preceding problem to turn the identity on this factor into alternating terms multiplying by $\exp(\pm 4\pi ix)$ – starting with $-$. Now, you are on $H \oplus l_2(H)$, 'renumbering' allows you to regard this as $l_2(H)$ again and when you do so your curve has become alternate multiplication by $\exp(\pm 4\pi ix)$ (with + first). Finally then, apply the preceding problem again, to deform to the identity (always of course through closed curves). Presto, Eilenberg's swindle!

Department of Mathematics, Massachusetts Institute of Technology