18.102 Introduction to Functional Analysis
Spring 2009

PROBLEM SET 4 FOR 18.102, SPRING 2009 DUE 11AM TUESDAY 10 MAR.

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Just to compensate for last week, I will make this problem set too short and easy!

1. Problem 4.1

Let H be a normed space in which the norm satisfies the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2) \ \forall \ u, v \in H. \tag{1}$$

Show that the norm comes from a positive definite sesquilinear (i.e. Hermitian) inner product. Big Hint:- Try

$$(u, v) = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2\right)! \tag{2}$$

2. Problem 4.2

Let H be a finite dimensional (pre)Hilbert space. So, by definition H has a basis $\{v_i\}_{i=1}^n$, meaning that any element of H can be written

$$v = \sum_i c_i v_i \tag{1}$$

and there is no dependence relation between the v_i's – the presentation of $v = 0$ in the form (1) is unique. Show that H has an orthonormal basis, $\{e_i\}_{i=1}^n$ satisfying $(e_i, e_j) = \delta_{ij}$ ($= 1$ if $i = j$ and 0 otherwise). Check that for the orthonormal basis the coefficients in (1) are $c_i = (v, e_i)$ and that the map

$$T : H \ni v \longmapsto ((v, e_i)) \in \mathbb{C}^n \tag{2}$$

is a linear isomorphism with the properties

$$(u, v) = \sum_i (Tu)_i \overline{(Tv)_i}, \ \|u\|_H = \|Tu\|_{\mathbb{C}^n} \ \forall \ u, v \in H. \tag{3}$$

Why is a finite dimensional preHilbert space a Hilbert space?

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