Lecture 9

We quickly review the definition of measure zero.

A set $A \subseteq \mathbb{R}^n$ is of *measure zero* if for every $\epsilon > 0$, there exists a covering of A by rectangles $Q_1, Q_2, Q_3, \ldots$ such that the total volume $\sum v(Q_i) < \epsilon$.

Remark. In this definition we can replace "rectangles" by "open rectangles." To see this, given any $\epsilon > 0$ let $Q_1, Q_2, \ldots$ be a cover of A with volume less than $\epsilon/2$. Next, choose Q_i' to be rectangles such that $\text{Int } Q_i' \supset Q_i$ and $v(Q_i') < 2v(Q_i)$. Then $\text{Int } Q_1', \text{Int } Q_2', \ldots$ cover A and have total volume less than ϵ.

We also review the three properties of measure zero that we mentioned last time, and we prove the third.

1. Let $A, B \subseteq \mathbb{R}^n$ and suppose $B \subset A$. If A is of measure zero, then B is also of measure zero.
2. Let $A_i \subseteq \mathbb{R}^n$ for $i = 1, 2, 3, \ldots$, and suppose the A_i's are of measure zero. Then $\cup A_i$ is also of measure zero.
3. Rectangles are *not* of measure zero.

We prove the third property:

Claim. *If Q is a rectangle, then Q is* not *of measure zero.*

Proof. Choose $\epsilon < v(Q)$. Suppose $Q_1, Q_2, \ldots$ are rectangles such that the total volume is less than ϵ and such that $\text{Int } Q_1, \text{Int } Q_2, \ldots$ cover Q.

The set Q is compact, so the H-B Theorem implies that the collection of sets $\text{Int } Q_1, \ldots, \text{Int } Q_N$ cover Q for N sufficiently large. So,

$$Q \subseteq \bigcup_{i=1}^{N} Q_i, \tag{3.36}$$

which implies that

$$v(Q) \leq \sum_{i=1}^{N} v(Q_i) < \epsilon < v(Q), \tag{3.37}$$

which is a contradiction. □

We then have the following simple result.

Claim. *If* $\text{Int } A$ *is non-empty, then A is* not *of measure zero.*

Proof. Consider any $p \in \text{Int } A$. There exists a $\delta > 0$ such that $U(p, \delta) = \{x : |x - p| < \delta\}$ is contained in A. Then let $Q = \{x : |x - p| \leq \delta\}$. It follows that if A is of measure zero, then Q is of measure zero, by the first property. We know that Q is not of measure zero by the third property. □

We restate the necessary and sufficient condition for R. integrability from last time, and we now prove the theorem.

Theorem 3.11. *Let Q be a rectangle and $f : Q \to \mathbb{R}$ be a bounded function. Let D be the set of points in Q where f is not continuous. Then f is R. integrable if and only if D is of measure zero.*

Proof. First we show that

$$D \text{ is of measure zero} \implies f \text{ is R. integrable} \tag{3.38}$$

Lemma 3.12. *Let $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$, and let $Q^\alpha, \alpha = 1, \dots, N$, be a covering of Q by rectangles. Then there exists a partition P of Q such that every rectangle R belonging to P is contained in Q^α for some α.*

Proof. Write out $Q^\alpha = I_1^\alpha \times \cdots \times I_n^\alpha$, and let

$$P_j = \left(\bigcup_\alpha \text{Endpoints of } I_j^\alpha \right) \cap [a_j, b_j] \cup \{a_j, b_j\}. \tag{3.39}$$

One can show that P_j is a partition of $[a_j, b_j]$, and $P = (P_1, \dots, P_n)$ is a partition of Q with the above properties. □

Let $f : Q \to \mathbb{R}$ be a bounded function, and let D be the set of points at which f is discontinuous. Assume that D is of measure zero. We want to show that f is R. integrable.

Let $\epsilon > 0$, and let $Q_i', i = 1, 2, 3, \dots$ be a collection of rectangles of total volume less than ϵ such that Int $Q_1', Q_2', \dots$ cover D.

If $p \in Q - D$, we know that f is continuous at p. So, there exists a rectangle Q_p with $p \in \text{Int } Q_p$ and $|f(x) - f(p)| < \epsilon/2$ for all $x \in Q_p$ (for example, $Q_p = \{x | |x - p| \leq \delta\}$ for some δ). Given any $x, y \in Q_p$, we find that $|f(x) - f(y)| < \epsilon$.

The rectangles Int $Q_p, p \in Q - D$ along with the rectangles Int $Q_i', i = 1, 2, \dots$ cover Q. The set Q is compact, so the H-B Theorem implies that there exists a finite open subcover:

$$Q_i \equiv \text{Int } Q_{p_i}, i = 1, \dots, \ell; \quad \text{Int } Q_j', j = 1, \dots, \ell. \tag{3.40}$$

Using the lemma, there exists a partition P of Q such that every rectangle belonging to P is contained in a Q_i or a Q_j'.

We now show that f is R. integrable.

$$\begin{aligned} U(f, P) - L(f, P) = &\sum_R (M_R(f) - m_R(f)) v(R) \\ &+ \sum_{R'} (M_{R'}(f) - m_{R'}(f)) v(R'), \end{aligned} \tag{3.41}$$

where each R in the first sum belongs to a Q_i, and each R' in the second sum belongs to a Q'_j.

We look at the first sum. If $x, y \in R \subseteq Q_i$, then $|f(x) - f(y)| \leq \epsilon$. So, $M_R(f) - m_R(f) \leq \epsilon$. It follows that

$$\begin{aligned} \sum_R (M_R(f) - m_R(f))v(R) &\leq \epsilon \sum_R v(R) \\ &\leq \epsilon v(Q). \end{aligned} \tag{3.42}$$

We now look at the second sum. The function $f : Q \to \mathbb{R}$ is bounded, so there exists a number c such that $-c \leq f(x) \leq c$ for all $x \in Q$. Then, $M_{R'}(f) - m_{R'}(f) \leq 2c$ so

$$\begin{aligned} \sum_{R'} (M_{R'}(f) - f_{R'}(f))v(R') &\leq 2c \sum_{R'} v(R') \\ &= 2c \sum_{i=1}^{\ell} \sum_{R' \subseteq Q'_i} v(R') \\ &\leq 2c \sum_i v(Q'_i) \\ &\leq 2c\epsilon. \end{aligned} \tag{3.43}$$

Substituting back into Equation 3.41, we get

$$U(f, P) - L(f, P) \leq \epsilon(v(Q) + 2c). \tag{3.44}$$

So,

$$\overline{\int_Q} f - \underline{\int_Q} f \leq \epsilon(v(Q) + 2c), \tag{3.45}$$

because

$$U(f, P) \geq \overline{\int_Q} f \text{ and } L(f, P) \leq \underline{\int_Q} f. \tag{3.46}$$

Letting ϵ go to zero, we conclude that

$$\overline{\int_Q} f = \underline{\int_Q} f, \tag{3.47}$$

which shows that f is Riemann integrable.

This concluses the proof in one direction. We do not prove the other direction. □

Corollary 4. *Suppose $f : Q \to \mathbb{R}$ is R. integrable and that $f \geq 0$ everywhere. If $\int_Q f = 0$, then $f = 0$ except on a set of measure zero.*

Proof. Let D be the set of points where f is discontinuous. The function f is R. integrable, so D is of measure zero.

If $p \in Q - D$, then $f(p) = 0$. To see this, suppose that $f(p) = \delta > 0$. The function f is continuous at p, so there exists a rectangle R_0 centered at p such that $f \geq n\delta/2$ on R_0. Choose a partition P such that R_0 is a rectangle belonging to P. On any rectangle R belonging to P, $f \geq 0$, so $m_R(f) \geq 0$. This shows that

$$\begin{aligned} L(f, P) &= m_{R_0}(f)v(R_0) + \sum_{R \neq R_0} m_R(f)v(R) \\ &\geq \frac{\delta}{2}v(R_0) + 0. \end{aligned} \tag{3.48}$$

But we assumed that $\int_Q f = 0$, so we have reached a contradiction. So $f = 0$ at all points $p \in Q - D$. $\square$