Lecture 34

6.6 Orientation of Manifolds

Let X be an n-dimensional manifold in $\mathbb{R}^N$. Assume that X is a closed subset of $\mathbb{R}^N$. Let $f : X \to \mathbb{R}$ be a $\mathcal{C}^\infty$ map.

Definition 6.21. We remind you that the support of f is defined to be

$$\text{supp } f = \overline{\{x \in X : f(x) \neq 0\}}. \tag{6.69}$$

Since X is closed, we don't have to worry about whether we are taking the closure in X or in $\mathbb{R}^n$.

Note that

$$f \in \mathcal{C}_0^\infty(X) \iff \text{supp } f \text{ is compact}. \tag{6.70}$$

Let $\omega \in \Omega^k(X)$. Then

$$\text{supp } \omega = \overline{\{p \in X : \omega_p \neq 0\}}. \tag{6.71}$$

We use the notation

$$\omega \in \Omega_c^k(X) \iff \text{supp } \omega \text{ is compact}. \tag{6.72}$$

We will be using partitions of unity, so we remind you of the definition:

Definition 6.22. A collection of functions $\{\rho_i \in \mathcal{C}_0^\infty(X) : i = 1, 2, 3, \dots\}$ is a *partition of unity* if

1. $0 \leq \rho_i$,
2. For every compact set $A \subseteq X$, there exists $N > 0$ such that supp $\rho_i \cap A = \phi$ for all $i > N$,
3. $\sum \rho_i = 1$.

Suppose the collection of sets $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ is a covering of X by open subsets U_α of X.

Definition 6.23. The partition of unity $\rho_i,\ i = 1, 2, 3, \dots$, is *subordinate to* $\mathcal{U}$ if for every i, there exists $\alpha \in I$ such that supp $\rho_i \subseteq U_\alpha$.

Claim. *Given a collection of sets $\mathcal{U} = \{U_\alpha : \alpha \in I\}$, there exists a partition of unity subordinate to $\mathcal{U}$.*

Proof. For each $\alpha \in I$, let $\tilde{U}_\alpha$ be an open set in $\mathbb{R}^N$ such that $U_\alpha = \tilde{U}_\alpha \cap X$. We define the collection of sets $\tilde{\mathcal{U}} = \{\tilde{U}_\alpha : \alpha \in I\}$. Let

$$\tilde{U} = \bigcup \tilde{U}_\alpha. \tag{6.73}$$

From our study of Euclidean space, we know that there exists a partition of unity $\tilde{\rho}_i \in \mathcal{C}_0^\infty(\tilde{U}),\ i = 1, 2, 3, \ldots$, subordinate to $\tilde{\mathcal{U}}$. Let $\iota_X : X \to \tilde{\mathcal{U}}$ be the inclusion map. Then

$$\rho_i = \tilde{\rho}_i \circ \iota_X = \iota_X^* \tilde{\rho}_i, \tag{6.74}$$

which you should check. □

We review orientations in Euclidean space before generalizing to manifolds. For a more comprehensive review, read section 7 of the Multi-linear Algebra notes.

Suppose $\mathbb{L}$ is a one-dimensional vector space and that $v \in \mathbb{L} - \{0\}$. The set $\mathbb{L} - \{0\}$ has two components:

$$\{\lambda v : \lambda > 0\} \text{ and } \{\lambda v : \lambda < 0\}. \tag{6.75}$$

Definition 6.24. An *orientation of* $\mathbb{L}$ is a choice of one of these components.

Notation. We call the preferred component $\mathbb{L}_+$ (the positive component). We call the other component $\mathbb{L}_-$ (the negative component).

We define a vector v to be *positively oriented* if $v \in \mathbb{L}_+$.

Now, let V be an n-dimensional vector space.

Definition 6.25. An *orientation of* V is an orientation of the one-dimensional vector space $\Lambda^n(V^*)$. That is, an orientation of V is a choice of $\Lambda^n(V^*)_+$.

Suppose that V_1, V_2 are oriented n-dimensional vector spaces, and let $A : V_1 \to V_2$ be a bijective linear map.

Definition 6.26. The map A is *orientation preserving* if

$$\omega \in \Lambda^n(V_2)_+ \implies A^*\omega \in \Lambda^n(V_1)_+. \tag{6.76}$$

Suppose that V_3 is also an oriented n-dimensional vector space, and let $B : V_2 \to V_3$ be a bijective linear map. If A and B are orientation preserving, then BA is also orientation preserving.

Finally, let us generalize the notion of orientation to orientations of manifolds. Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold.

Definition 6.27. An *orientation of* X is a function on X which assigns to each point $p \in X$ an orientation of T_pX.

We give two examples of orientations of a manifold:

Example 1: Let $\omega \in \Lambda^n(X)$, and suppose that ω is nowhere vanishing. Orient X by assigning to $p \in X$ the orientation of T_pX for which $\omega_p \in \Lambda^n(T_p^*X)_+$.

Example 2: Take $X = U$, an open subset of $\mathbb{R}^n$, and let

$$\omega = dx_1 \wedge \cdots \wedge dx_n. \tag{6.77}$$

Define an orientation as in the first example. This orientation is called the *standard orientation of* U.

Definition 6.28. An orientation of X is a $\mathcal{C}^\infty$ *orientation* if for every point $p \in X$, there exists a neighborhood U of p in X and an n-form $\omega \in \Omega^n(U)$ such that for all points $q \in U$, $\omega_q \in \Lambda^n(T_q^*X)_+$.

From now on, we will only consider $\mathcal{C}^\infty$ orientations.

Theorem 6.29. *If X is oriented, then there exists $\omega \in \Omega^n(X)$ such that for all $p \in X$, $\omega_p \in \Lambda^n(T_p^*X)_+$.*

Proof. For every point $p \in X$, there exists a neighborhood U_p of p and an n- form $\omega^{(p)} \in \Omega^n(U_p)$ such that for all $q \in U_p$, $(\omega^{(p)})_q \in \Lambda^n(T_Q^*X)_+$.

Take $\rho_i,\ i = 1, 2, \ldots$, a partition of unity subordinate to $\mathcal{U} = \{U_p : p \in X\}$. For every i, there exists a point p such that $\rho_i \in \mathcal{C}_0^\infty(U_p)$. Let

$$\omega_i = \begin{cases} \rho_i \omega^{(p)} & \text{on } U_p, \\ 0 & \text{on the } X - U_p. \end{cases} \tag{6.78}$$

Since the ρ_i's are compactly supported, ω_i is a $\mathcal{C}^\infty$ map. Let

$$\omega = \sum \omega_i. \tag{6.79}$$

One can check that ω is positively oriented at every point. □

Definition 6.30. An n-form $\omega \in \Omega^n(X)$ with the property hypothesized in the above theorem is called a *volume form*.

Remark. If ω_1, ω_2 are volume forms, then we can write $\omega_2 = f\omega_1$, for some $f \in \mathcal{C}^\infty(X)$ (where $f \neq 0$ everywhere). In general, $f(p) > 0$ because $(\omega_1)_p, (\omega_2)_p \in \Lambda^n(T_p^*X)_+$. So, if ω_1, ω_2 are volume forms, then $\omega_2 = f\omega_1$, for some $f \in \mathcal{C}^\infty(X)$ such that $f > 0$.

Remark. Problem #6 on the homework asks you to show that if X is orientable and connected, then there are exactly two ways to orient it. This is easily proved using the above Remark.

Suppose that $X \subseteq \mathbb{R}^n$ is a one-dimensional manifold (a "curve"). Then T_pX is one-dimensional. We can find vectors $v, -v \in T_pX$ such that $||v|| = 1$. An orientation of X is just a choice of v or $-v$.

Now, suppose that X is an $(n-1)$-dimensional manifold in $\mathbb{R}^n$. Define

$$N_pX = \{v \in T_p\mathbb{R}^n : v \perp w \text{ for all } w \in T_pX\}. \tag{6.80}$$

Then $\dim N_pX = 1$, so you can find $v, -v \in N_pX$ such that $||v|| = 1$. By Exercise #5 in section 4 of the Multi-linear Algebra Notes, an orientation of T_pX is just a choice of v or $-v$.

Suppose X_1, X_2 are oriented n-dimensional manifolds, and let $f : X_1 \to X_2$ be a diffeomorphism.

Definition 6.31. The map f is *orientation preserving* if for every $p \in X_1$,

$$df_p : T_pX_1 \to T_qX_2 \tag{6.81}$$

is orientation preserving, where $q = f(p)$.

Remark. Let ω_2 be a volume form on X_2. Then f is orientation preserving if and only if $f^*\omega_2 = \omega_1$ is a volume form on X_1.

We look at an example of what it means for a map to be orientation preserving. Let U, V be open sets on $\mathbb{R}^n$ with the standard orientation. Let $f : U \to V$ be a diffeomorphism. So, by definition, the form

$$dx_1 \wedge \cdots \wedge dx_n \tag{6.82}$$

is a volume form of V. The form

$$f^*dx_1 \wedge \cdots \wedge dx_n = \det \left[\frac{\partial f_i}{\partial x_j}\right] dx_1 \wedge \cdots \wedge dx_n \tag{6.83}$$

is a volume form of U if and only if

$$\det \left[\frac{\partial f_i}{\partial x_j}\right] > 0, \tag{6.84}$$

that is, if and only if f is orientation preserving in our old sense.

Now that we have studied orientations of manifolds, we have all of the ingredients we need to study integration theory for manifolds.