Lecture 2

1.6 Compactness

As usual, throughout this section we let (X, d) be a metric space. We also remind you from last lecture we defined the open set

$$U(x_o, \lambda) = \{x \in X : d(x, x_o) < \lambda\}. \tag{1.10}$$

Remark. If $U(x_o, \lambda) \subseteq U(x_1, \lambda_1)$, then $\lambda_1 > d(x_o, x_1)$.

Remark. If $A_i \subseteq U(x_o, \lambda_i)$ for $i = 1, 2$, then $A_1 \cup A_2 \subseteq U(x_o, \lambda_1 + \lambda_2)$.

Before we define compactness, we first define the notions of boundedness and covering.

Definition 1.19. A subset A of X is *bounded* if $A \subseteq U(x_o, \lambda)$ for some λ.

Definition 1.20. Let $A \subseteq X$. A collection of subsets $\{U_\alpha \subseteq X, \alpha \in I\}$ is a *cover* of A if

$$A \subset \bigcup_{\alpha \in I} U_\alpha.$$

Now we turn to the notion of compactness. First, we only consider compact sets as subsets of $\mathbb{R}^n$.

For any subset $A \subseteq \mathbb{R}^n$,

$$A \text{ is compact} \iff A \text{ is closed and bounded.}$$

The above statement holds true for $\mathbb{R}^n$ but not for general metric spaces. To motivate the definition of compactness for the general case, we give the Heine-Borel Theorem.

Heine-Borel (H-B) Theorem. *Let $A \subseteq \mathbb{R}^n$ be compact and let $\{U_\alpha, \alpha \in I\}$ be a cover of A by open sets. Then a finite number of U_α's already cover A.*

The property that a finite number of the U_α's cover A is called the Heine-Borel (H-B) property*. So, the H-B Theorem can be restated as follows: If A is compact in $\mathbb{R}^n$, then A has the H-B property.*

Sketch of Proof. First, we check the H-B Theorem for some simple compact subsets of $\mathbb{R}^n$. Consider rectangles $Q = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$, where $I_k = [a_k, b_k]$ for each k. Starting with one dimension, it can by shown by induction that these rectangles have the H-B property.

Too prove the H-B theorem for general compact subsets, consider any closed and bounded (and therefore compact) subset A of $\mathbb{R}^n$. Since A is bounded, there exists a rectangle Q such that $A \subseteq Q$. Suppose that the collection of subsets $\{U_\alpha, \alpha \in I\}$ is an open cover of A. Then, define $U_o = \mathbb{R}^n - A$ and include U_o in the open cover. The rectangle Q has the H-B property and is covered by this new cover, so there exists a finite subcover covering Q. Furthermore, the rectangle Q contains A, so the finite subcover also covers A, proving the H-B Theorem for general compact subsets.

□

The following theorem further motivates the general definition for compactness.

Theorem 1.21. *If $A \subseteq \mathbb{R}^n$ has the H-B property, then A is compact.*

Sketch of Proof. We need to show that the H-B property implies A is bounded (which we leave as an exercise) and closed (which we prove here).

To show that A is closed, it is sufficient to show that A^c is open. Take any $x_o \in A^c$, and define

$$C_N = \{x \in \mathbb{R}^n : d(x, x_o) \leq 1/N\}, \tag{1.11}$$

and

$$U_N = C_N^c. \tag{1.12}$$

Then,

$$\bigcap C_N = \{x_o\} \tag{1.13}$$

and

$$\bigcup U_N = \mathbb{R}^n - \{x_o\}. \tag{1.14}$$

The U_N's cover A, so the H-B Theorem implies that there is a finite subcover $\{U_{N_1}, \ldots, U_{N_k}\}$ of A. We can take $N_1 < N_2 < \cdots < N_k$, so that $A \subseteq U_{N_k}$. By taking the complement, it follows that $C_{N_k} \subseteq A^c$. But $U(x_o, 1/N_k) \subseteq C_{N_k}$, so x_o is contained in an open subset of A^c. The above holds for any $x_o \in A^c$, so A^c is open. □

Let us consider the above theorem for arbitrary metric space (X, d) and $A \subseteq X$.

Theorem 1.22. *If $A \subseteq X$ has the H-B property, then A is closed and bounded.*

Sketch of Proof. The proof is basically the same as for the previous theorem. □

Unfortunately, the converse is not always true. Finally, we come to our general definition of compactness.

Definition 1.23. A subset $A \subseteq X$ is *compact* if it has the H-B property.

Compact sets have many useful properties, some of which we list here in the theorems that follow.

Theorem 1.24. *Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous map. If A is a compact subset of X, then $f(A)$ is a compact subset of Y.*

Proof. Let $\{U_\alpha, \alpha \in I\}$ be an open covering of $f(A)$. Each pre-image $f^{-1}(U_\alpha)$ is open in X, so $\{f^{-1}(U_\alpha) : \alpha \in I\}$ is an open covering of A. The H-B Theorem says that there is a finite subcover $\{f^{-1}(U_{\alpha_i}) : 1 \le i \le N\}$. It follows that the collection $\{U_{\alpha_i} : 1 \le i \le N\}$ covers $f(A)$, so $f(A)$ is compact. □

A special case of the above theorem proves the following theorem.

Theorem 1.25. *Let A be a compact subset of X and $f : X \to \mathbb{R}$ be a continuous map. Then f has a maximum point on A.*

Proof. By the above theorem, $f(A)$ is compact, which implies that $f(a)$ is closed and and bounded. Let $a =$ l.u.b. of $f(a)$. The point a is in $f(A)$ because $f(A)$ is closed, so there exists an $x_o \in A$ such that $f(x_o) = a$. □

Another useful property of compact sets involves the notion of uniform continuity.

Definition 1.26. Let $f : X \to \mathbb{R}$ be a continuous function, and let A be a subset of X. The map f is *uniformly continuous on* A if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon,$$

for all $x, y \in A$.

Theorem 1.27. *If $f : X \to Y$ is continuous and A is a compact subset of X, then f is uniformly continuous on A.*

Proof. Let $p \in A$. There exists a $\delta_p > 0$ such that $|f(x) - f(p)| < \epsilon/2$ for all $x \in U(p, \delta_p)$. Now, consider the collection of sets $\{U(p, \delta_p/2) : p \in A\}$, which is an open cover of A. The H-B Theorem says that there is a finite subcover $\{U(p_i, \delta_{p_i}/2) : 1 \le i \le N\}$. Choose $\delta \le \min \delta_{p_i}/2$. The following claim finishes the proof.

Claim. *If $d(x, y) < \delta$, then $|f(x) - f(y)| < \epsilon$.*

Proof. Given x, choose p_i such that $x \in U(p_i, \delta_{p_i}/2)$. So, $d(p_i, x) < \delta_{p_i}/2$ and $d(x, y) < \delta < \delta_{p_i}/2$. By the triangle inequality we conclude that $d(p_i, y) < \delta_{p_i}$. This shows that $x, y \in U(p_i, \delta_{p_i})$, which implies that $|f(x) - f(p_i)| < \epsilon/2$ and $|f(y) - f(p_i)| < \epsilon/2$. Finally, by the triangle inequality, $|f(x) - f(y)| < \epsilon$, which proves our claim.

□

1.7 Connectedness

As usual, let (X, d) be a metric space.

Definition 1.28. The metric space (X, d) is *connected* if it is impossible to write X as a disjoint union $X = U_1 \cup U_2$ of non-empty open sets U_1 and U_2.

Note that disjoint simply means that $U_1 \cap U_2 = \phi$, where ϕ is the empty set.

A few simple examples of connected spaces are $\mathbb{R}$, $\mathbb{R}^n$, and $I = [a, b]$. The following theorem shows that a connected space gets mapped to a connected subspace by a continuous function.

Theorem 1.29. *Given metric spaces (X, d_X) and (Y, d_Y), and a continuous map $f : X \to Y$, it follows that*

$$X \text{ is connected } \implies f(X) \text{ is connected.}$$

Proof. Suppose $f(X)$ can be written as a union of open sets $f(X) = U_1 \cup U_2$ such that $U_1 \cap U_2 = \phi$. Then $X = f^{-1}(U_1) \cup f^{-1}(U_2)$ is a disjoint union of open sets. This contradicts that X is connected. □

The intermediate-value theorem follows as a special case of the above theorem.

Intermediate-value Theorem. *Let (X, d) be connected and $f : X \to \mathbb{R}$ be a continuous map. If $a, b \in f(X)$ and $a < r < b$, then $r \in f(X)$.*

Proof. Suppose $r \notin f(X)$. Let $A = (-\infty, r)$ and $B = (r, \infty)$. Then $X = f^{-1}(A) \cup f^{-1}(B)$ is a disjoint union of open sets, a contradiction. □