

Lecture 1

1 Review of Topology

1.1 Metric Spaces

Definition 1.1. Let X be a set. Define the Cartesian product $X \times X = \{(x, y) : x, y \in X\}$.

Definition 1.2. Let $d : X \times X \rightarrow \mathbb{R}$ be a mapping. The mapping d is a *metric on* X if the following four conditions hold for all $x, y, z \in X$:

- (i) $d(x, y) = d(y, x)$,
- (ii) $d(x, y) \geq 0$,
- (iii) $d(x, y) = 0 \iff x = y$, and
- (iv) $d(x, z) \leq d(x, y) + d(y, z)$.

Given a metric d on X , the pair (X, d) is called a *metric space*.

Suppose d is a metric on X and that $Y \subseteq X$. Then there is an automatic metric d_Y on Y defined by restricting d to the subspace $Y \times Y$,

$$d_Y = d|_{Y \times Y}. \tag{1.1}$$

Together with Y , the metric d_Y defines the automatic metric space (Y, d_Y) .

1.2 Open and Closed Sets

In this section we review some basic definitions and propositions in topology. We review open sets, closed sets, norms, continuity, and closure. Throughout this section, we let (X, d) be a metric space unless otherwise specified.

One of the basic notions of topology is that of the *open set*. To define an open set, we first define the ϵ -neighborhood.

Definition 1.3. Given a point $x_o \in X$, and a real number $\epsilon > 0$, we define

$$U(x_o, \epsilon) = \{x \in X : d(x, x_o) < \epsilon\}. \tag{1.2}$$

We call $U(x_o, \epsilon)$ the ϵ -neighborhood of x_o in X .

Given a subset $Y \subseteq X$, the ϵ -neighborhood of x_o in Y is just $U(x_o, \epsilon) \cap Y$.

Definition 1.4. A subset U of X is *open* if for every $x_o \in U$ there exists a real number $\epsilon > 0$ such that $U(x_o, \epsilon) \subseteq U$.

We make some propositions about the union and intersections of open sets. We omit the proofs, which are fairly straightforward.

The following Proposition states that arbitrary unions of open sets are open.

Proposition 1.5. *Let $\{U_\alpha, \alpha \in I\}$ be a collection of open sets in X , where I is just a labeling set that can be finite or infinite. Then, the set*

$$\bigcup_{\alpha \in I} U_\alpha \text{ is open.}$$

The following Corollary is an application of the above Proposition.

Corollary 1. *If $Y \subset X$ and A is open in Y (w.r.t. d_Y), then there exists an open set U in X such that $U \cap Y = A$.*

Proof. The set A is open in Y . So, for any $p \in A$ there exists an $\epsilon_p > 0$ such that $U(p, \epsilon_p) \cap Y \subseteq A$. We construct a set U containing A by taking the union of the sets $U(p, \epsilon_p)$ over all p in A ,

$$U = \bigcup_{p \in A} U(p, \epsilon_p). \tag{1.3}$$

For every $p \in A$, we have $U(p, \epsilon_p) \cap Y \subseteq A$, which shows that $U \cap Y \subseteq A$. Furthermore, the union is over all $p \in A$, so $A \subseteq U$, which implies that $A \subseteq U \cap Y$. This shows that $U \cap Y = A$. To conclude the proof, we see that U is open by the openness of the $U(p, \epsilon_p)$ and the above theorem. \square

The following Proposition states that finite intersections of open sets are open.

Proposition 1.6. *Let $\{U_i, i = 1, \dots, N\}$ be a finite collection of open sets in X . Then the set*

$$\bigcap_{i=1}^{i=N} U_i \text{ is open.}$$

Definition 1.7. Define the *complement of A in X* to be $A^c = X - A = \{x \in X : x \notin A\}$.

We use the complement to define *closed* sets.

Definition 1.8. The set A is *closed in X* if A^c is open in X .

1.3 Metrics on \mathbb{R}^n

For most of this course, we will only consider the case $X = \mathbb{R}^n$ or X equals certain subsets of \mathbb{R}^n called manifolds, which we will define later.

There are two interesting metrics on \mathbb{R}^n . They are the *Euclidean metric* and the *sup metric*, and are defined in terms of the *Euclidean norm* and the *sup norm*, respectively.

Definition 1.9. Let $x \in \mathbb{R}^n$, written out in component form as $x = (x_1, x_2, \dots, x_n)$. The *Euclidean norm* of x is

$$\|x\| = \sqrt{x_1^2 + \dots + x_n^2},$$

and the *sup norm* of x is

$$|x| = \max_i |x_i|.$$

From these norms we obtain the Euclidean distance function

$$\|x - y\| \tag{1.4}$$

and the sup distance function

$$|x - y|, \tag{1.5}$$

respectively.

These two distance functions are related in several ways. In particular,

$$|x - y| \leq \|x - y\| \leq \sqrt{n}|x - y|.$$

These distance functions are also related by the following Proposition, which will sometimes come in handy.

Proposition 1.10. *A subset U of \mathbb{R}^n is open w.r.t. the $\| \cdot \|$ distance function if and only if it is open w.r.t. the $| \cdot |$ distance function.*

So, these two distance functions give the same topologies of \mathbb{R}^n .

1.4 Continuity

Consider two metric spaces (X, d_X) and (Y, d_Y) , a function $f : X \rightarrow Y$, and a point $x_o \in X$.

Definition 1.11. The function f is *continuous at x_o* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(x, x_o) < \delta \implies d_Y(f(x), f(x_o)) < \epsilon. \tag{1.6}$$

By definition, a function is continuous if it is continuous at all points in its domain.

Definition 1.12. The function f is *continuous* if f is continuous at every point $x_o \in X$.

There is an alternative formulation of continuity that we present here as a theorem.

Theorem 1.13. *The function f is continuous if and only if for every open subset U of Y , the pre-image $f^{-1}(U)$ is open in X .*

Continuous functions can often be combined to construct other continuous functions. For example, if $f, g : X \rightarrow \mathbb{R}$ are continuous functions, then $f + g$ and fg are continuous functions.

1.5 Limit Points and Closure

As usual, let (X, d) be a metric space.

Definition 1.14. Suppose that $A \subseteq X$. The point $x_o \in X$ is a *limit point* of A if for every ϵ -neighborhood $U(x_o, \epsilon)$ of x_o , the set $U(x_o, \epsilon)$ is an infinite set.

Definition 1.15. The *closure* of A , denoted by \bar{A} , is the union of A and the set of limit points of A ,

$$\bar{A} = A \cup \{x_o \in X : x_o \text{ is a limit point of } A\}. \quad (1.7)$$

Now we define the interior, exterior, and the boundary of a set in terms of open sets. In the following, we denote the complement of A by $A^c = X - A$.

Definition 1.16. The set

$$\text{Int } A \equiv (\bar{A}^c)^c \quad (1.8)$$

is called the *interior* of A .

It follows that

$$x \in \text{Int } A \iff \exists \epsilon > 0 \text{ such that } U(x, \epsilon) \subset A. \quad (1.9)$$

Note that the interior of A is open.

We define the exterior of a set in terms of the interior of the set.

Definition 1.17. The *exterior* of A is defined to be $\text{Ext } A \equiv \text{Int } A^c$.

The boundary of a set is the collection of all points not in the interior or exterior.

Definition 1.18. The *boundary* of A is defined to be $\text{Bd } A \equiv X - ((\text{Ext } A) \cup (\text{Int } A))$.

Always, we have $X = \text{Int } A \cup \text{Ext } A \cup \text{Bd } A$.