Multi-linear Algebra
Notes for 18.101

1 Linear algebra

To read these notes you will need some background in linear algebra. In particular you'll need to be familiar with the material in § 1–2 of Munkres and § 1 of Spivak. In this section we will discuss a couple of items which are frequently, but not always, covered in linear algebra courses, but which we'll need for our "crash course" in multilinear algebra in § 2–6.

The quotient spaces of a vector space

Let V be a vector space and W a vector subspace of V. A *W-coset* is a set of the form

$$v + W = \{v + w \, , \, w \in W\} \, .$$

It is easy to check that if $v_1 - v_2 \in W$, the cosets, $v_1 + W$ and $v_2 + W$, coincide while if $v_1 - v_2 \notin W$, they are disjoint. Thus the W-cosets decompose V into a *disjoint* collection of subsets of V. We will denote this collection of sets by V/W.

One defines a vector addition operation on V/W by defining the sum of two cosets, $v_1 + W$ and $v_2 + W$ to be the coset

$$v_1 + v_2 + W \tag{1.1}$$

and one defines a scalar multiplication operation by defining the scalar multiple of $v + W$ by λ to be the coset

$$\lambda v + W \, . \tag{1.2}$$

It is easy to see that these operations are well defined. For instance, suppose $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$. Then $v_1 - v_1'$ and $v_2 - v_2'$ are in W; so $(v_1 + v_2) - (v_1' + v_2')$ is in W and hence $v_1 + v_2 + W = v_1' + v_2' + W$.

These operations make V/W into a vector space, and one calls this space the *quotient space* of V by W.

We define a mapping

$$\pi : V \to V/W \tag{1.3}$$

by setting $\pi(v) = v + W$. It's clear from (1.1) and (1.2) that π is a linear mapping. Moreover, for every coset, $v + W$, $\pi(v) = v + W$; so the mapping, π, is onto. Also note that the zero vector in the vector space, V/W, is the zero coset, $0+W=W$. Hence v is in the kernel of π if $v+W=W$, i.e., $v\in W$. In other words the kernel of π is W.

In the definition above, V and W don't have to be finite dimensional, but if they are, then one can show

$$\dim V/W = \dim V - \dim W\,. \tag{1.4}$$

(A proof of this is sketched in exercises 1–3.)

The dual space of a vector space

We'll denote by V^* the set of all linear functions, $\ell : V \to \mathbb{R}$. If ℓ_1 and ℓ_2 are linear functions, their sum, $\ell_1+\ell_2$, is linear, and if ℓ is a linear function and λ is a real number, the function, $\lambda\ell$, is linear. Hence V^* is a vector space. One calls this space the *dual space* of V.

Suppose V is n-dimensional, and let $e_1,\dots,e_n$ be a basis of V. Then every vector, $v\in V$, can be written uniquely as a sum

$$v = c_1v_1+\cdots+c_nv_n \quad c_i\in\mathbb{R}\,.$$

Let

$$e_i^*(v) = c_i\,. \tag{1.5}$$

If $v=c_1e_1+\cdots+c_ne_n$ and $v'=c_1'e_1+\cdots+c_n'e_n$ then $v+v'=(c_1+c_1')e_1+\cdots+(c_n+c_n')e_n$, so

$$e_i^*(v+v') = c_i+c_i' = e_i^*(v)+e_i^*(v')\,.$$

This shows that $e_i^*(v)$ is a linear function of v and hence $e_i^*\in V^*$.

Claim: e_i^*, $i=1,\dots,n$ is a basis of V^*.

Proof. First of all note that by (1.5)

$$e_i^*(e_j) = \begin{cases} 1\,, & i=j \\ 0\,, & i\neq j \end{cases}\,. \tag{1.6}$$

If $\ell\in V^*$ let $\lambda_i=\ell(e_i)$ and let $\ell'=\sum\lambda_ie_i^*$. Then by (1.6)

$$\ell'(e_j) = \sum\lambda_ie_i^*(e_j) = \lambda_j = \ell(e_j)\,, \tag{1.7}$$

i.e., ℓ and ℓ' take identical values on the basis vectors, e_j. Hence $\ell=\ell'$.

Suppose next that $\sum\lambda_ie_i^*=0$. Then by (1.6), with $\ell'=0$; $\lambda_j=0$. Hence the e_j^*'s are linearly independent.

□

Let V and W be vector spaces and

$$A : V \to W \tag{1.8}$$

a linear map. Given $\ell \in W^*$, the composition $\ell \circ A$ of A with the linear map $\ell : W \to \mathbb{R}$ is linear, and hence is an element of V^*. We will denote this element by $A^*\ell$, and we will denote by

$$A^* : W^* \to V^*$$

the map, $\ell \to A^*\ell$. It's clear from the definition that

$$A^*(\ell_1 + \ell_2) = A^*\ell_1 + A^*\ell_2$$

and that

$$A^*\lambda\ell = \lambda A^*\ell\,,$$

i.e., that A^* is linear.

Definition. A^* is the transpose of the mapping A.

We will conclude this section by giving a matrix description of A^*. Let $e_1, \ldots, e_n$ be a basis of V and $f_1, \ldots, f_m$ a basis of W; let $e_1^*, \ldots, e_n^*$ and $f_1^*, \ldots, f_m^*$ be the dual bases of V^* and W^*. Suppose A is defined in terms of $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ by the $m \times n$ matrix, $[a_{i,j}]$, i.e., suppose

$$Ae_j = \sum a_{i,j} f_i\,.$$

Claim. A^* is defined, in terms of $f_1^*, \ldots, f_m^*$ and $e_1^*, \ldots, e_n^*$ by the transpose matrix, $[a_{j,i}]$.

Proof. Let

$$A^* f_i^* = \sum c_{j,i} e_j^*\,.$$

Then

$$A^* f_i^*(e_j) = \sum_k c_{k,i} e_k^*(e_j) = c_{j,i}$$

by (1.5). On the other hand

$$\begin{aligned} A^* f_i^*(e_j) &= f_i^*(Ae_j) \\ &= \sum_k a_{j,k} f_i^*(f_k) = a_{j,i}\,, \end{aligned}$$

so $a_{j,i} = c_{j,i}$.

$\square$

Exercises.

1. Let V be an n-dimensional vector space and W a k-dimensional subspace. Show that there exists a basis, $e_1, \ldots, e_n$ of V with the property that $e_1, \ldots, e_k$ is a basis of W. *Hint:* Induction on $n-k$. To start the induction suppose that $n-k=1$. Let $e_1, \ldots, e_{n-1}$ be a basis of W and e_n any vector in $V-W$.

2. In exercise 1 show that the vectors $f_i = \pi(e_{k+i})$, $i = 1, \ldots, n-k$ are a basis of V/W. Conclude that the dimension of V/W is $n-k$.

3. In exercise 1 let U be the linear span of the vectors, e_{k+i}, $i = 1, \ldots, n-k$.

   Show that the map

   $$U \to V/W\,, \quad u \to \pi(u)\,, \tag{1.9}$$

   is a vector space isomorphism, i.e., show that it maps U bijectively onto V/W.[1]

4. Let U, V and W be vector spaces and let $A : V \to W$ and $B : U \to V$ be linear mappings. Show that $(AB)^* = B^* A^*$.

[1]This exercise shows that the notion of "quotient space", which can be somewhat daunting when one first encounters it, is in essence no more complicated than the notion of "subspace".

2 Tensors

Let V be an n-dimensional vector space and let V^k be the set of all k-tuples, $(v_1, \dots, v_k)$, $v_i \in V$. A function

$$T : V^k \to \mathbb{R}$$

is said to be linear in its i^{th} variable if, when we fix vectors, $v_1, \dots, v_{i-1}$, $v_{i+1}, \dots, v_k$, the map

$$v \in V \to T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k) \tag{2.1}$$

is linear in V. If T is linear in its i^{th} variable for $i = 1, \dots, k$ it is said to be *k-linear*, or alternatively is said to be a *k-tensor*. We denote the set of all k-tensors by $\mathcal{L}^k(V)$.

Let T_1 and T_2 be functions on V^k. It is clear from (2.1) that if T_1 and T_2 are k-linear, so is $T_1 + T_2$. Similarly if T is k-linear and λ is a real number, λT is k-linear. Hence $\mathcal{L}^k(V)$ is a vector space. Note that for $k = 1$, "k-linear" just means "linear", so $\mathcal{L}^1(V) = V^*$.

We will next prove that this vector space is finite dimensional. Let

$$I = (i_1, \dots i_k)$$

be a sequence of integers with $1 \le i_r \le n$, $r = 1, \dots, k$. We will call such a sequence *a multi-index* of length k. For instance the multi-indices of length 2 are the square arrays of pairs of integers

$$(i, j)\,,\ 1 \le i, j \le n$$

and there are exactly n^2 of them.

Exercise.

Show that there are exactly n^k multi-indices of length k.

Now fix a basis, $e_1, \dots, e_n$, of V and for $T \in \mathcal{L}^k(V)$ let

$$T_I = T(e_{i_1}, \dots, e_{i_k}) \tag{2.2}$$

for every multi-index of length k, I.

Proposition 2.1. *The T_I's determine T, i.e., if T and T' are k-tensors and $T_I = T'_I$ for all I, then $T = T'$.*

Proof. By induction on n. For $n = 1$ we proved this result in § 1. Let's prove that if this assertion is true for $n - 1$, it's true for n. For each e_i let T_i be the $(k-1)$-tensor mapping

$$(v_1, \dots, v_{n-1}) \to T(v_1, \dots, v_{n-1}, e_i)\,.$$

Then for $v = c_1 e_1 + \cdots c_n e_n$,

$$T(v_1, \ldots, v_{n-1}, v) = \sum c_i T_i(v_1, \ldots, v_{n-1}),$$

so the T_i's determine T. Now apply induction.

$\square$

The tensor product operation

If T_1 is a k-tensor and T_2 is an ℓ-tensor, one can define a $k+\ell$-tensor, $T_1 \otimes T_2$, by setting

$$(T_1 \otimes T_2)(v_1, \ldots, v_{k+\ell}) = T_1(v_1, \ldots, v_k) T_2(v_{k+1}, \ldots, v_{k+\ell}).$$

This tensor is called *the tensor product* of T_1 and T_2. Similarly, given a k-tensor, T_1, an ℓ-tensor T_2 and an m-tensor T_3, one can define a $(k+\ell+m)$-tensor $T_1 \otimes T_2 \otimes T_3$ by setting

$$\begin{aligned} T_1 \otimes T_2 \otimes T_3(v_1, \ldots, v_{k+\ell}) \\ = T_1(v_1, \ldots, v_k) T_2(v_{k+1}, \ldots, v_{k+\ell}) T_3(v_{k+\ell+1}, \ldots, v_{k+\ell+m}). \end{aligned} \tag{2.3}$$

Alternatively, one can define (2.3) by defining it to be the tensor product of $T_1 \otimes T_2$ and T_3 or the tensor product of T_1 and $T_2 \otimes T_3$. It's easy to see that both these tensor products are identical with (2.3):

$$(T_1 \otimes T_2) \otimes T_3 = T_1 \otimes (T_2 \otimes T_3) = T_1 \otimes T_2 \otimes T_3. \tag{2.4}$$

We leave for you to check that if λ is a real number

$$\lambda(T_1 \otimes T_2) = (\lambda T_1) \otimes T_2 = T_1 \otimes (\lambda T_2) \tag{2.5}$$

and that the left and right distributive laws are valid: For $k_1 = k_2$,

$$(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3 \tag{2.6}$$

and for $k_2 = k_3$

$$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3. \tag{2.7}$$

A particularly interesting tensor product is the following. For $i = 1, \ldots, k$ let $\ell_i \in V^*$ and let

$$T = \ell_1 \otimes \cdots \otimes \ell_k. \tag{2.8}$$

Thus, by definition,

$$T(v_1, \ldots, v_k) = \ell_1(v_1) \ldots \ell_k(v_k). \tag{2.9}$$

A tensor of the form (2.9) is called a *decomposible* k-tensor. These tensors, as we will see, play an important role in what follows. In particular, let $e_1, \dots, e_n$ be a basis of V and $e_1^*, \dots, e_n^*$ the dual basis of V^*. For every multi-index I of length k let

$$e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^* .$$

Then if J is another multi-index of length k,

$$e_I^*(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1\,, & I = J \\ 0\,, & I \neq J \end{cases} \tag{2.10}$$

by (1.6), (2.8) and (2.9). From (2.10) it's easy to conclude

Theorem 2.2. *The e_I^*'s are a basis of $\mathcal{L}^k(V)$.*

Proof. Given $T \in \mathcal{L}^k(V)$, let

$$T' = \sum T_I e_I^*$$

where the T_I's are defined by (2.2). Then

$$T'(e_{j_1}, \dots, e_{j_k}) = \sum T_I e_I^*(e_{j_1}, \dots, e_{j_k}) = T_J \tag{2.11}$$

by (2.10); however, by Proposition 2.1 the T_J's determine T, so $T' = T$. This proves that the e_I^*'s are a spanning set of vectors for $\mathcal{L}^k(V)$. To prove they're a basis, suppose

$$\sum C_I e_I^* = 0$$

for constants, $C_I \in \mathbb{R}$. Then by (2.11) with $T = 0$, $C_J = 0$, so the e_I^*'s are linearly independent.

□

As we noted above there are exactly n^k multi-indices of length k and hence n^k basis vectors in the set, $\{e_I^*\}$, so we've proved

Corollary. $\dim \mathcal{L}^k(V) = n^k$.

The pull-back operation

Let V and W be finite dimensional vector spaces and let $A : V \to W$ be a linear mapping. If $T \in \mathcal{L}^k(W)$, we define

$$A^*T : V^k \to \mathbb{R}$$

to be the function

$$A^*T(v_1, \dots, v_k) = T(Av_1, \dots, Av_k) \,. \tag{2.12}$$

It's clear from the linearity of A that this function is linear in its i^{th} variable for all i, and hence is a k-tensor. We will call A^*T the *pull-back* of T by the map, A.

Proposition 2.3. *The map*

$$A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)\,, \quad T \to A^*T\,, \tag{2.13}$$

is a linear mapping.

We leave this as an exercise. We also leave as an exercise the identity

$$A^*(T_1 \otimes T_2) = A^*T_1 \otimes A^*T_2 \tag{2.14}$$

for $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^m(W)$. Also, if U is a vector space and $B : U \to V$ a linear mapping, we leave for you to check that

$$(AB)^*T = B^*(A^*T) \tag{2.15}$$

for all $T \in \mathcal{L}^k(W)$.

Exercises.

1. Verify that there are exactly n^k multi-indices of length k.
2. Prove Proposition 2.3.
3. Verify (2.14).
4. Verify (2.15).

3 Alternating k-tensors

We will discuss in this section a class of k-tensors which play an important role in multivariable calculus. In this discussion we will need some standard facts about the "permutation group". For those of you who are already familiar with this object (and I suspect most of you are) you can regard the paragraph below as a chance to re-familiarize yourselves with these facts.

Permutations

Let $\sum_k$ be the k-element set: $\{1, 2, \ldots, k\}$. A *permutation of order* k is a bijective map, $\sigma : \sum_k \to \sum_k$, Given two permutations, σ_1 and σ_2, their *product*, $\sigma_1\sigma_2$, is the composition of σ_1 and σ_2, i.e., the map,

$$i \to \sigma_2(\sigma_1(i))\,,$$

and for every permutation, σ, one denotes by σ^{-1} the inverse permutation:

$$\sigma(i) = j \Leftrightarrow \sigma^{-1}(j) = i\,.$$

Let S_k be the set of all permutations of order k. One calls S_k the *permutation group* of $\sum_k$ or, alternatively, the *symmetric group on* k *letters.*

Check:

There are $k!$ elements in S_k.

For every $1 \le i < j \le k$, let $\tau = \tau_{i,j}$ be the permutation

$$\begin{aligned} \tau(i) &= j \\ \tau(j) &= i \\ \tau(\ell) &= \ell\,, \quad \ell \neq i, j\,. \end{aligned} \tag{3.1}$$

τ is called a *transposition*, and if $j = i + 1$, τ is called an *elementary transposition.*

Theorem 3.1. *Every permutation can be written as a product of finite numbers of transpositions.*

Proof. Induction on k: "$k = 2$" is obvious. *The induction step: "$k - 1$" implies "k":* Given $\sigma \in S_k$, $\sigma(k) = i \Leftrightarrow \sigma\tau_{ik}(k) = k$. Thus $\sigma\tau_{ik}$ is, in effect, a permutation of $\sum_{k-1}$. By induction, $\sigma\tau_{ik}$ can be written as a product of transpositions, so

$$\sigma = (\sigma\tau_{ik})\tau_{ik}$$

can be written as a product of transpositions.

□

Theorem 3.2. *Every transposition can be written as a product of elementary transpositions.*

Proof. Let $\tau = \tau_{ij}$, $i < j$. With i fixed, argue by induction on j. Note that

$$\tau_{ij} = \tau_{j-1,j}\tau_{i,j-1}\tau_{j-1,j}\,.$$

Now apply induction to $\tau_{i,j-1}$.

$\square$

The sign of a permutation

Let $x_1, \ldots, x_k$ be the coordinate functions on $\mathbb{R}^k$. For $\sigma \in S_k$ we define

$$(-1)^\sigma = \prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}\,. \tag{3.2}$$

Notice that the numerator and denominator in this expression are identical up to sign. Indeed, if $p = \sigma(i) < \sigma(j) = q$, the term, $x_p - x_q$ occurs once and just once in the numerator and once and just once in the denominator; and if $q = \sigma(i) > \sigma(j) = p$, the term, $x_p - x_q$, occurs once and just once in the numerator and its negative, $x_q - x_p$, occurs once and just once in the numerator. Thus

$$(-1)^\sigma = \pm 1\,. \tag{3.3}$$

Claim:

For $\sigma, \tau \in S_k$

$$(-1)^{\sigma\tau} = (-1)^\sigma (-1)^\tau\,. \tag{3.4}$$

Proof. By definition,

$$(-1)^{\sigma\tau} = \prod_{i<j} \frac{x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{x_i - x_j}\,.$$

We write the right hand side as a product of

$$\prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j} = (-1)^\sigma \tag{3.5}$$

and

$$\prod_{i<j} \frac{x_{\sigma\tau(i)} - x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}} \tag{3.6}$$

For $i < j$, let $p = \sigma(i)$ and $q = \sigma(j)$ when $\sigma(i) < \sigma(j)$ and let $p = \sigma(j)$ and $q = \sigma(i)$ when $\sigma(j) < \sigma(i)$. Then

$$\frac{x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{x_{\sigma(i)} - x_{\sigma(j)}} = \frac{x_{\tau(p)} - x_{\tau(q)}}{x_p - x_q}$$

(i.e., if $\sigma(i) < \sigma(j)$, the numerator and denominator on the right *equal* the numerator and denominator on the left and, if $\sigma(j) < \sigma(i)$ are *negatives* of the numerator and denominator on the left). Thus (3.2) becomes

$$\prod_{p<q} \frac{x_{\tau(p)} - x_{\tau(q)}}{x_p - x_q} = (-1)^\tau .$$

$\square$

We'll leave for you to check that if τ is a transposition, $(-1)^\tau = -1$ and to conclude from this:

Proposition 3.3. *If σ is the product of an odd number of transpositions, $(-1)^\sigma = -1$ and if σ is the product of an even number of transpositions $(-1)^\sigma = +1$.*

Alternation

Let V be an n-dimensional vector space and $T \in \mathcal{L}^k(v)$ a k-tensor. If $\sigma \in S_k$, let $T^\sigma \in \mathcal{L}^k(V)$ be the k-tensor

$$T^\sigma(v_1, \dots, v_k) = T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(n)}) . \tag{3.7}$$

Proposition 3.4.

1. *If $T = \ell_1 \otimes \cdots \otimes \ell_k$, $\ell_i \in V^*$ then $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$.*
2. *The map, $T \in \mathcal{L}^k(V) \to T^\sigma \in \mathcal{L}^k(V)$ is a linear map.*
3. *$(T^\sigma)^\tau = T^{\tau\sigma}$.*

Proof. To prove 1, we note that by (3.7)

$$\begin{aligned} &(\ell_1 \otimes \cdots \otimes \ell_k)^\sigma(v_1, \dots, v_k) \\ &= \ell_1(v_{\sigma^{-1}(1)}) \cdots \ell_k(v_{\sigma^{-1}(k)}) . \end{aligned}$$

Setting $\sigma^{-1}(i) = q$, the i^{th} term in this product is $\ell_{\sigma(q)}(v_q)$; so the product can be rewritten as

$$\ell_{\sigma(1)}(v_1) \dots \ell_{\sigma(k)}(v_k)$$

or

$$(\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)})(v_1, \dots, v_k) .$$

The proof of 2 we'll leave as an exercise.

Proof of 3: By item 2, it suffices to check 3 for decomposible tensors. However, by 1

$$\begin{aligned}(\ell_1 \otimes \cdots \otimes \ell_k)^{\sigma\tau} &= \ell_{\sigma\tau(1)} \otimes \cdots \otimes \ell_{\sigma\tau(k)} \\ &= (\ell_{\tau(1)} \otimes \cdots \otimes \ell_{\tau(k)})^{\sigma} .\end{aligned}$$

Definition 3.5. *$T \in \mathcal{L}^k(V)$ is* alternating *if $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.*

We will denote by $\mathcal{A}^k(V)$ the set of all alternating k-tensors in $\mathcal{L}^k(V)$. By item 2 of Proposition 3.4 this set is a vector subspace of $\mathcal{L}^k(V)$.

It is not easy to write down simple examples of alternating k-tensors; however, there is a method, called the *alternation operation*, for constructing such tensors: Given $T \in \mathcal{L}^k(V)$ let

$$\operatorname{Alt}(T) = \sum_{\tau \in S_k} (-1)^\tau T^\tau . \tag{3.8}$$

We claim

Proposition 3.6. *For $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$,*

1. $\operatorname{Alt}(T)^\sigma = (-1)^\sigma \operatorname{Alt}(T)$
2. *if* $T \in \mathcal{A}^k(V)$, $\operatorname{Alt}(T) = k!T$.
3. $\operatorname{Alt}(T^\sigma) = \operatorname{Alt}(T)^\sigma$
4. *the map*
$$\operatorname{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V) ,\ T \to \operatorname{Alt}(T)$$
is linear.

Proof. To prove 1 we note:

$$\begin{aligned}\operatorname{Alt} T^\sigma &= \sum (-1)^\tau (T^{\tau\sigma}) \\ &= (-1)^\sigma \sum (-1)^{\tau\sigma} T^{\tau\sigma} .\end{aligned}$$

But as τ runs over S_k, $\tau\sigma$ runs over S_k, and hence the right hand side is $(-1)^\sigma \operatorname{Alt}(T)$. ☐

Proof of 2. If $T \in \mathcal{A}^k$,

$$\begin{aligned}\operatorname{Alt}(T) &= \sum (-1)^\tau T^\tau \\ &= \sum (-1)^\tau (-1)^\tau T \\ &= k!\,T .\end{aligned}$$

☐

Proof of 3.

$$\begin{aligned}\operatorname{Alt}(T^\sigma) &= \sum(-1)^\tau T^{\tau\sigma} = (-1)^\sigma \sum (-1)^{\tau\sigma} T^{\tau\sigma} \\ &= (-1)^\sigma \operatorname{Alt}(T) = \operatorname{Alt}(T)^\sigma \,.\end{aligned}$$

□

Finally, item 4 is an easy corollary of item 2 of Proposition 3.4.

□

We will use this alternation operation to construct a basis for $\mathcal{A}^k(V)$. First, however, we require some notation:

Let $I = (i_1, \ldots, i_k)$ be a multi-index of length k.

Definition 3.7. *1. I is* repeating *if $i_r = i_s$ for some $r \neq s$.*

2. I is strictly increasing if $i_1 < i_2 < \cdots < i_r$.

3. For $\sigma \in S_k$, $I^\sigma = \left(i_{\sigma(1)}, \ldots, i_{\sigma(k)}\right)$.

Remark: If I is non-repeating there is a unique $\sigma \in S_k$ so that I^σ is strictly increasing.

Let $e_1, \ldots, e_n$ be a basis of V and let

$$e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$$

and

$$\psi_I = \operatorname{Alt}(e_I^*) \,.$$

Proposition 3.8. *1. $\psi_{I^\sigma} = (-1)^\sigma \psi_I$.*

2. If I is repeating, $\psi_I = 0$.

3. If I and J are strictly increasing,

$$\psi_I(e_{j_1}, \ldots, e_{j_k}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases} \quad .$$

Proof. To prove 1 we note that $(e_I^*)^\sigma = e_{I^\sigma}^*$; so

$$\operatorname{Alt}(e_{I^\sigma}^*) = \operatorname{Alt}(e_I^*)^\sigma = (-1)^\sigma \operatorname{Alt}(e_I^*) \,.$$

□

Proof of 2: Suppose $I = (i_1, \ldots, i_k)$ with $i_r = i_s$ for $r \neq s$. Then if $\tau = \tau_{i_r, i_s}$, $e_I^* = e_{I^r}^*$ so

$$\psi_I = \psi_{I^r} = (-1)^\tau \psi_I = -\psi_I \,.$$

□

Proof of 3: By definition

$$\psi_I(e_{j_1},\dots,e_{j_k}) = \sum (-1)^\tau e^*_{I^\tau}(e_{j_1},\dots,e_{j_k})\,.$$

But by (2.10)

$$e^*_{I^\tau}(e_{j_1},\dots,e_{j_k}) = \left\{\begin{array}{l} 1 \text{ if } I^\tau = J \\ 0 \text{ if } I^\tau \neq J \end{array}\right. \,. \tag{3.9}$$

Thus if I and J are strictly increasing, I^τ is strictly increasing if and only if $I^\tau = I$, and (3.9) is zero if and only if $I = J$.

□

Now let $T \in \mathcal{A}^k$. Then by Proposition 2.2,

$$T = \sum a_J e^*_J\,, \quad a_J \in \mathbb{R}\,.$$

Since

$$\begin{aligned} k!T &= \operatorname{Alt}(T), \\ T &= \frac{1}{k!}\sum a_J \operatorname{Alt}(e^*_J) = \sum b_J \psi_J\,. \end{aligned}$$

We can discard all repeating terms in this sum since they are zero; and for every non-repeating term, J, we can write $J = I^\sigma$, where I is strictly increasing, and hence $\psi_J = (-1)^\sigma \psi_I$.

Conclusion:

We can write T as a sum

$$T = \sum c_I \psi_I\,, \tag{3.10}$$

with I's strictly increasing.

Claim.

The c_I's are unique.

Proof. For J strictly increasing

$$T(e_{j_1},\dots,e_{j_k}) = \sum c_I \psi_I(e_{j_1},\dots,e_{j_k}) = c_J\,. \tag{3.11}$$

By (3.10) the ψ_I's, I strictly increasing, are a spanning set of vectors for $\mathcal{A}^k(V)$, and by (3.11) they are linearly independent, so we've proved

Proposition 3.9. *The alternating tensors, ψ_I, I strictly increasing, are a basis for $\mathcal{A}^k(V)$.*

Thus $\dim \mathcal{A}^k(V)$ is equal to the number of strictly increasing multi-indices, I, of length k. We leave for you as an exercise to show that this number is equal to

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \text{“}n \text{ choose } k\text{”} \tag{3.12}$$

if $1 \leq k \leq n$.

$\square$

Hint: Show that every strictly increasing multi-index of length k determines a k element subset of $\{1, \ldots, n\}$ and vice-versa.

Note also that if $k > n$ every multi-index

$$I = (i_1, \ldots, i_k)$$

of length k has to be repeating: $i_r = i_s$ for some $r \neq s$ since the i_p's lie on the interval $1 \leq i \leq n$. Thus by Proposition 3.8

$$\psi_I = 0$$

for all multi-indices of length $k > 0$ and

$$\mathcal{A}^k = \{0\}\,. \tag{3.13}$$

Exercises.

1. Show that there are exactly $k!$ permutations of order k. *Hint:* Induction on k: Let $\sigma \in S_k$, and let $\sigma(k) = i$, $1 \leq i \leq k$. Show that $\sigma\tau_{ik}$ leaves k fixed and hence is, in effect, a permutation of $\sum_{k-1}$.

2. Prove that if $\tau \in S_k$ is a transposition, then $(-1)^\tau = -1$.

3. Prove assertion 2 in Proposition 3.4.

4. Prove that $\dim \mathcal{A}^k(V)$ is given by (3.13).

5. Verify that for $i < j - 1$,

   $$\tau_{i,j} = \tau_{j-1,j}\tau_{i,j-1}, \tau_{j-1,j}\,.$$

6. For $k = 3$ show that every one of the six elements of S_3 is either a transposition or can be written as a product of two transpositions.

7. Let $\sigma \in S_k$ be the “cyclic” permutation

   $$\sigma(i) = i + 1\,, \quad i = 1, \ldots, k - 1$$

   and $\sigma(k) = 1$. Show explicitly how to write σ as a product of transpositions and compute $(-1)^\sigma$. *Hint:* Same hint as in exercise 1.

4 The space, $\Lambda^k(V^*)$

In § 3 we showed that the image of the alternation operation, Alt : $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$ is $\mathcal{A}^k(V)$. In this section we will compute the kernel of Alt .

Definition 4.1. *A decomposible k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$, $\ell_i \in V^*$, is redundant if for some index, i, $\ell_i = \ell_{i+1}$.*

Let $\mathcal{L}^k$ be the linear span of the set of redundant k-tensors.

Note that for $k = 1$ the notion of redundant doesn't really make sense; a single vector $\ell \in \mathcal{L}^1(V^*)$ can't be "redundant" so we decree

$$\mathcal{I}^1(V) = \{0\}\,.$$

Proposition 4.2. *If $T \in \mathcal{I}^k$, then* $\mathrm{Alt}\,(T) = 0$.

Proof. Let $T = \ell_k \otimes \cdots \otimes \ell_k$ with $\ell_i = \ell_{i+1}$. Then if $\tau = \tau_{i,i+1}$, $T^\tau = T$ and $(-1)^\tau = -1$. Hence $\mathrm{Alt}\,(T) = \mathrm{Alt}\,(T^\sigma) = \mathrm{Alt}\,(T)^\sigma = -\mathrm{Alt}\,(T)$; so $\mathrm{Alt}\,(T) = 0$. □

To simplify notation let's abbreviate $\mathcal{L}^k(V)$, $\mathcal{A}^k(V)$ and $\mathcal{I}^k(V)$ by $\mathcal{L}^k$, $\mathcal{A}^k$ and $\mathcal{I}^k$.

Proposition 4.3. *If $T \in \mathcal{I}^r$ and $T' \in \mathcal{L}^s$ then $T \otimes T'$ and $T' \otimes T$ are in $\mathcal{I}^{r+s}$.*

Proof. We can assume that T and T' are decomposible, i.e., $T = \ell_1 \otimes \cdots \otimes \ell_r$ and $T' = \ell_1' \otimes \cdots \otimes \ell_s'$ and that T is redundant: $\ell_i = \ell_{i+1}$. Then

$$T \otimes T' = \ell_1 \otimes \cdots \ell_{i-1} \otimes \ell_i \otimes \ell_i \otimes \cdots \ell_r \otimes \ell_1' \otimes \cdots \otimes \ell_s'$$

is redundant and hence in $\mathcal{I}^{r+s}$. The argument for $T' \otimes T$ is similar. □

Proposition 4.4. *If $T \in \mathcal{L}^k$ and $\sigma \in S_k$, then*

$$T^\sigma = (-1)^\sigma T + S \tag{4.1}$$

where S is in $\mathcal{I}^k$.

Proof. We can assume T is decomposible, i.e., $T = \ell_1 \otimes \cdots \otimes \ell_k$. Let's first look at the simplest possible case: $k = 2$ and $\sigma = \tau_{1,2}$. Then

$$\begin{aligned} T^\sigma - (-1)^\sigma T &= \ell_1 \otimes \ell_2 + \ell_2 \otimes \ell_1 \\ &= ((\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2)/2\,, \end{aligned}$$

and the right hand side is decomposible, and hence in $\mathcal{I}^2$. Next let k be arbitrary and $\sigma = \tau_{i,i+1}$. If $T_1 = \ell_1 \otimes \cdots \otimes \ell_{i-2}$ and $T_2 = \ell_{i+2} \otimes \cdots \otimes \ell_k$. Then

$$T - (-1)^\sigma T = T_1 \otimes (\ell_i \otimes \ell_{i+1} + \ell_{i+1} \otimes \ell_i) \otimes T_2$$

is in $\mathcal{I}^k$ by Proposition 4.3 and the computation above.

The general case: By Theorem 3.2, σ can be written as a product of m elementary transpositions, and we'll prove (4.1) by induction on m.

We've just dealt with the case $m = 1$.

The induction step: "$m-1$" implies "m". Let $\sigma = \tau\beta$ where β is a product of $m-1$ elementary transpositions and τ is an elementary transposition. Then

$$\begin{aligned} T^\sigma = (T^\beta)^\tau &= (-1)^\tau T^\beta + \cdots \\ &= (-1)^\tau (-1)^\beta T + \cdots \\ &= (-1)^\sigma T + \cdots \end{aligned}$$

where the "dots" are elements of $\mathcal{I}^k$, and the induction hypothesis was used in line 2.

□

Corollary. *If $T \in \mathcal{L}^k$, then*

$$\mathrm{Alt}\,(T) = k!T + W\,, \tag{4.2}$$

where W is in $\mathcal{I}^k$.

Proof. By definition $\mathrm{Alt}\,(T) = \sum (-1)^\sigma T^\sigma$, and by Proposition 4.4, $T^\sigma = (-1)^\sigma T + W_\sigma$, with $W_\sigma \in \mathcal{I}^k$. Thus,

$$\begin{aligned} \mathrm{Alt}\,(T) &= \sum (-1)^\sigma (-1)^\sigma T + \sum (-1)^\sigma W_\sigma \\ &= k!T + W \end{aligned}$$

where $W = \sum (-1)^\sigma W_\sigma$.

□

Corollary. *$\mathcal{I}^k$ is the kernel of* Alt .

Proof. We've already proved that if $T \in \mathcal{I}^k$, $\mathrm{Alt}\,(T) = 0$. To prove the converse assertion we note that if $\mathrm{Alt}\,(T) = 0$, then by (4.2)

$$T = -\tfrac{1}{k!} W\,.$$

with $W \in \mathcal{I}^k$. □

Putting these results together we conclude:

Theorem 4.5. *Every element, T, of $\mathcal{L}^k$ can be written uniquely as a sum, $T = T_1 + T_2$ where $T_1 \in \mathcal{A}^k$ and $T_2 \in \mathcal{I}^k$.*

Proof. By (4.2), $T = T_1 + T_2$ with

$$T_1 = \frac{1}{k!} \operatorname{Alt}(T)$$

and

$$T_2 = -\frac{1}{k!} W \,.$$

To prove that this decomposition is unique, suppose $T_1 + T_2 = 0$, with $T_1 \in \mathcal{A}^k$ and $T_2 \in \mathcal{I}^k$. Then

$$0 = \operatorname{Alt}(T_1 + T_2) = k! T_1$$

so $T_1 = 0$, and hence $T_2 = 0$.

□

Let

$$\Lambda^k(V^*) = \mathcal{L}^k(V^*)/\mathcal{I}^k(V^*) \,, \tag{4.3}$$

i.e., let $\Lambda^k = \Lambda^k(V^*)$ be the quotient of the vector space $\mathcal{L}^k$ by the subspace $\mathcal{I}^k$ of $\mathcal{L}^k$. By (1.3) one has a linear map:

$$\pi : \mathcal{L}^k \to \Lambda^k \,, \quad T \to T + \mathcal{I}^k \tag{4.4}$$

which is onto and has $\mathcal{I}^k$ as kernel. We claim:

Theorem 4.6. *The map, π, maps $\mathcal{A}^k$ bijectively onto Λ^k.*

Proof. By Theorem 4.5 every $\mathcal{I}^k$ coset, $T + \mathcal{I}^k$, contains a unique element T_1 of $\mathcal{A}^k$. Hence, for every element of Λ^k there is a unique element of $\mathcal{A}^k$ which gets mapped onto it by π.

□

Remark. Since Λ^k and $\mathcal{A}^k$ are isomorphic as vector spaces many treatments of multilinear algebra avoid mentioning Λ^k, reasoning that $\mathcal{A}^k$ is a perfectly good substitute for it and that one should, if possible, not make two different definitions for what is essentially the same object. This is a justifiable point of view (and is the point of view taken by Spivak and Munkres[2]). There are, however, some advantages to distinguishing between A^k and Λ^k, as we'll see in § 5.

Exercises.

1. A k-tensor $T \in \mathcal{L}^k(V)$ is *symmetric* if $T^\sigma = T$ for all $\sigma \in S_k$. Show that the set $\mathcal{S}^k(V)$ of symmetric k tensors is a vector subspace of $\mathcal{L}^k(V)$.

[2] and by Guillemin–Pollack in their book, Differential Topology

2. Let $e_1, \ldots, e_n$ be a basis of V. Show that every symmetric 2-tensor is of the form

$$\sum a_{ij} e_i^* \otimes e_j^*,$$

where $q_{i,j} = a_{j,i}$ and $e_1^*, \ldots, e_n^*$ are the dual basis vectors of V^*.

3. Show that if T is a symmetric k-tensor, then for $k \geq 2$, T is in $\mathcal{I}^k$. *Hint:* Let σ be a transposition and deduce from the identity, $T^\sigma = T$, that T has to be in the kernel of Alt.

4. *Warning:* In general $\mathcal{S}^k(V) \neq \mathcal{I}^k(V)$. Show, however, that if $k = 2$ these two spaces are equal.

5. Conclude from exercise 4 that if $T \in \mathcal{I}^k$ then T can be written as a sum

$$\sum_{r=0}^{k} T_1^{(r)} \otimes T_2^{(r)} \otimes T_3^{(r)}$$

where $T_1^{(1)} \in \mathcal{L}^r(V)$, $T_2^{(r)} \in \mathcal{S}^2(V)$ and $T_3^{(r)} \in \mathcal{L}^{k-r-2}$.

5 The wedge product

The tensor algebra operations on the spaces $\mathcal{L}^k(V)$ which we discussed in Sections 2 and 3 (i.e., the "tensor product operation" and the "pull-back" operation), give rise to similar operations on the spaces Λ^k. We will discuss in this section the analogue of the tensor product operation. As in § 4 we'll abbreviate $\mathcal{L}^k(V)$ to $\mathcal{L}^k$ and $\Lambda^k(V)$ to Λ^k when it's clear which "V" is intended.

Given $\omega_i \in \Lambda^{k_i}$, where $i = 1, 2$, we can by (4.4) find $T_i \in \mathcal{L}^{k_i}$ with $\omega_i = \pi(T_i)$. Then $T_1 \otimes T_2 \in \mathcal{L}^{k_1+k_2}$. Let

$$\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2) \in \Lambda^{k_1+k_2}\,. \tag{5.1}$$

Claim.

This wedge product is well defined, i.e., doesn't depend on our choices of T_1 and T_2.

Proof. Let $\pi(T_1) = \pi(T_1') = \omega_1$. Then $T_1' = T_1 + W_1$ for some $W_1 \in \mathcal{I}^{k_1}$, so

$$T_1' \otimes T_2 = T_1 \otimes T_2 + W_1 \otimes T_2\,.$$

But $W_1 \in \mathcal{I}^{k_1}$ implies $W_1 \otimes T_2 \in \mathcal{I}^{k_1+k_2}$ and this implies:

$$\pi(T_1' \otimes T_2) = \pi(T_1 \otimes T_2)\,.$$

A similar argument shows that (5.1) is well-defined independent of the choice of T_2. □

More generally let $\omega_i \in \Lambda^{k_i}$, $i = 1, 2, 3$, and let $\omega_i = \pi(T_i)$, $T_i \in \mathcal{L}^{k_i}$. Define

$$\omega_1 \wedge \omega_2 \wedge \omega_3 \in \Lambda^{k_1+k_2+k_3}$$

by setting

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = \pi(T_1 \otimes T_2 \otimes T_3)\,.$$

As above it's easy to see that this is well-defined independent of the choice of T_1, T_2 and T_3. It is also easy to see that this triple wedge product is just the wedge product of $\omega_1 \wedge \omega_2$ with ω_3 or, alternatively, the wedge product of ω_1 with $\omega_2 \wedge \omega_3$, i.e.,

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = (\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3). \tag{5.2}$$

We leave for you to check:

For $\lambda \in \mathbb{R}$

$$\lambda(\omega_1 \wedge \omega_2) = (\lambda\omega_1) \wedge \omega_2 = \omega_1 \wedge (\lambda\omega_2) \tag{5.3}$$

and verify the two distributive laws:

(5.4)
$$(\omega_1 + \omega_2) \wedge \omega_3 \;=\; \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3$$

and

(5.5)
$$\omega_1 \wedge (\omega_2 + \omega_3) \;=\; \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3\,.$$

As we noted in § 4, $\mathcal{I}^k = \{0\}$ for $k = 1$, i.e., there are no non-zero "redundant" k tensors in degree $k = 1$. Thus

(5.6)
$$\Lambda^1(V^*) = V^* = \mathcal{L}^1(V^*).$$

A particularly interesting example of a wedge product is the following.

Let $\ell_i \in V^* = \Lambda^1(V^*)$, $i = 1, \ldots, k$. Then if $T = \ell_1 \otimes \cdots \otimes \ell_k$,

(5.7)
$$\ell_1 \wedge \cdots \wedge \ell_k = \pi(T) \in \Lambda^k(V^*)\,.$$

We will call (5.7) a *decomposible element* of $\Lambda^k(V^*)$.

We will next show that the permutation operation on k tensors

$$T \in \mathcal{L}^k \to T^\sigma \in \mathcal{L}^k$$

extends to a permutation operation on Λ^k. To see this we first note that if $T \in \mathcal{I}^k$, $T^\sigma \in \mathcal{I}^k$.

Proof. If $T = \ell_1 \otimes \cdots \otimes \ell_k$ is a redundant k-tensor, i.e., $\ell_i = \ell_{i+1}$, for some multi-index, i, and $\tau = \tau_{i,i+1}$, then $T = T^\tau$ and $(-1)^\tau = -1$ so

$$\begin{aligned} T^\sigma = T^{\tau\sigma} &= (-1)^{\tau\sigma} T + \cdots \\ &= (-1)^\tau (-1)^\sigma T + \cdots \\ &= = -T^\sigma + \cdots \end{aligned}$$

where the "dots" indicate elements of $\mathcal{I}^k$. Thus $T^\sigma \in \mathcal{I}^k$. Since every element of $\mathcal{I}^k$ is a sum of redundant tensors this proves the assertion above.

□

Now let ω be in Λ^k. Then $\omega = \pi(T)$ for some $T \in \mathcal{L}^k$, and we define

(5.8)
$$\omega^\sigma = \pi(T^\sigma)\,.$$

Claim:

This definition makes sense: ω^σ is well-defined.

Proof. Suppose $\omega = \pi(T) = \pi(T_1)$. Then $T_1 = T + W$, for some $W \in \mathcal{I}^k$ and $T_1^\sigma = T^\sigma + W^\sigma$. But W^σ is in $\mathcal{I}^k$, so

$$\omega^\sigma = \pi(T_1^\sigma) = \pi(T^\sigma)\,.$$

□

Example.

Let

$$\omega = \ell_1 \wedge \cdots \wedge \ell_k = \pi(\ell_1 \otimes \cdots \otimes \ell_k)$$

with $\ell_i \in V^*$. Then

$$\begin{aligned} \omega^\sigma &= \pi(\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}) \\ &= \ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} \,. \end{aligned} \tag{5.9}$$

We will next prove that

$$\omega^\sigma = (-1)^\sigma \omega \,. \tag{5.10}$$

Proof. Let $\omega = \pi(T)$ for $T \in \mathcal{L}^k$. Then

$$\omega^\sigma = \pi(T^\sigma) \,.$$

But

$$T^\sigma = (-1)^\sigma T + W \,,$$

for some W in $\mathcal{I}^k$, so

$$\omega^\sigma = \pi(T^\sigma) = (-1)^\sigma \pi(T) = (-1)^\sigma \omega \,.$$

□

Corollary. *For* $\ell_1, \ldots, \ell_k \in V^*$

$$\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k \,. \tag{5.11}$$

For instance,

$$\ell_1 \wedge \ell_2 = -\ell_2 \wedge \ell_1 \,. \tag{5.12}$$

Exercise:

Show that if $\omega_1 \in \Lambda^r$ and $\omega_2 \in \Lambda^s$ then

$$\omega_1 \wedge \omega_2 = (-1)^{rs} \omega_2 \wedge \omega_1 \,. \tag{5.13}$$

Hint: It suffices to prove this for decomposible elements, i.e., for

$$\omega_1 = \ell_1 \wedge \cdots \wedge \ell_r \,.$$

Now make rs applications of the formula (5.12).

Let $e_1, \ldots, e_n$ be a basis of V and let $e_1^*, \ldots, e_n^*$ be the dual basis of V^*. For every multi-index I of length k, let

$$\tilde{e}_I^* = e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* = \pi(e_I^*) \,. \tag{5.14}$$

Theorem 5.1. *The elements (5.14), with I strictly increasing, are basis vectors of Λ^k.*

Proof. The elements

$$\psi_I = \operatorname{Alt}\left(e_I^*\right), \; I \text{ strictly increasing,}$$

are basis vectors of $\mathcal{A}^k$ by Proposition 3.6; so their images, $\pi(\psi_I)$, are a basis of Λ^k. But

$$\begin{aligned} \pi(\psi_I) &= \pi\left(\sum(-1)^\sigma (e_I^*)^\sigma\right) \\ &= \sum(-1)^\sigma (\tilde{e}_I^*)^\sigma \\ &= \sum(-1)^\sigma (-1)^\sigma \tilde{e}_I^* \\ &= k!\tilde{e}_I^* \,. \end{aligned}$$

□

Exercises:

1. Prove the assertions (5.3), (5.4) and (5.6).

2. Verify the multiplication law, (5.13) for wedge product.

6 The pull-back operation on Λ^k

Let V and W be vector spaces and let A be a linear map of V into W. Given a k-tensor, $T \in \mathcal{L}^k(W)$, the *pull-back* A^*T is the k-tensor

$$A^*T(v_1, \dots, v_k) = T(Av_1, \dots, Av_k) \tag{6.1}$$

in $\mathcal{L}^k(V)$ (See § 2, equation 2.12). In this section we'll show how to define a similar pull-back operation on Λ^k.

Lemma 6.1. *If $T \in \mathcal{I}^k(W)$, then $A^*T \in \mathcal{I}^k(V)$.*

Proof. It suffices to verify this when T is a redundant k-tensor, i.e., a tensor of the form

$$T = \ell_1 \otimes \cdots \otimes \ell_k$$

where $\ell_r \in W^*$ and $\ell_i = \ell_{i+1}$ for some index, i. But by (2.14)

$$A^*T = A^*\ell_1 \otimes \cdots \otimes A^*\ell_k$$

and the tensor on the right is redundant since $A^*\ell_i = A^*\ell_{i+1}$.

□

Now let ω be an element of $\Lambda^k(W^*)$ and let $\omega = \pi(T)$ where T is in $\mathcal{L}^k(W)$. We define

$$A^*\omega = \pi(A^*T)\,. \tag{6.2}$$

Claim:

The left hand side of (6.2) is well-defined.

Proof. If $\omega = \pi(T) = \pi(T')$, then $T = T' + S$ for some $S \in \mathcal{I}^k(W)$, and $A^*T' = A^*T + A^*S$. But $A^*S \in \mathcal{I}^k(V)$, so

$$\pi(A^*T') = \pi(A^*T)\,.$$

Proposition 6.2. *(i) The map*

$$A^* : \Lambda^k(W^*) \to \Lambda^k(V^*)\,, \tag{6.3}$$

mapping ω to $A^*\omega$ is linear.

(ii) If $\omega_i \in \Lambda^{k_i}(W)$, $i = 1, 2$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*\omega_1 \wedge A^*\omega_2\,.$$

(iii) *If U is a vector space and $B : U \to V$ a linear map, then for $\omega \in \Lambda^k(W^*)$,*

$$B^* A^* \omega = (AB)^* \omega \,. \tag{6.4}$$

We'll leave the proof of these three assertions as exercises. *Hint:* They follow immediately from the analogous assertions for the pull-back operation on tensors (See (2.14) and (2,15)).

As an application of the pull-back operation we'll show how to use it to define the notion of *determinant* for a linear mapping. Let V be an n-dimensional vector space. Then $\dim \Lambda^n(V^*) = \binom{n}{n} = 1$; i.e., $\Lambda^n(V^*)$ is a *one-dimensional* vector space. Thus if $A : V \to V$ is a linear mapping, the induced pull-back mapping:

$$A^* : \Lambda^n(V^*) \to \Lambda^n(V^*) \,,$$

is just "multiplication by a constant". We denote this constant by $\det(A)$ and call it the *determinant* of A, Hence, by definition,

$$A^* \omega = \det(A) \omega \tag{6.5}$$

for all ω in $\Lambda^n(V^*)$. From (6.5) it's easy to derive a number of basic facts about determinants.

Proposition 6.3. *If A and B are linear mappings of V into V, then*

$$\det(AB) = \det(A) \det(B) \,. \tag{6.6}$$

Proof. By (6.4) and

$$\begin{aligned} (AB)^* \omega &= \det(AB) \omega \\ &= B^*(A^* \omega) = \det(B) A^* \omega \\ &= \det(B) \det(A) \omega \,, \end{aligned}$$

so, $\det(AB) = \det(A) \det(B)$.

□

Proposition 6.4. *If $I : V \to V$ is the identity map, $Iv = v$ for all $v \in V$, then $\det(I) = 1$.*

We'll leave the proof as an exercise. *Hint:* I^* is the identity map on $\Lambda^n(V^*)$.

Proposition 6.5. *If $A : V \to V$ is not onto, then $\det(A) = 0$.*

Proof. Let W be the image of A. Then if A is not onto, the dimension of W is less than n, so $\Lambda^n(W^*) = \{0\}$. Now let $A = I_W B$ where I_W is the inclusion map of W into V and B is the mapping A regarded as a mapping from V to W. Thus if ω is in $\Lambda^n(V^*)$, then by (6.4)

$$A^* \omega = B^* I_W^* \omega$$

and since $I_W^* \omega$ is in $\Lambda^n(W)$ it is zero.

□

$\square$

We will derive by wedge product arguments the familiar "matrix formula" for the determinant. Let V and W be n-dimensional vector spaces and let $e_1, \ldots, e_n$ be a basis for V and $f_1, \ldots, f_n$ a basis for W. From these bases we get dual bases, $e_1^*, \ldots, e_n^*$ and $f_1^*, \ldots, f_n^*$, for V^* and W^*. Moreover, if A is a linear map of V into W and $[a_{i,j}]$ the $n \times n$ matrix describing A in terms of these bases, then the transpose map, $A^* : W^* \to V^*$, is described in terms of these dual bases by the $n \times n$ transpose matrix, i.e., if

$$Ae_j = \sum a_{i,j} f_i \,,$$

then

$$A^* f_j^* = \sum a_{j,i} e_i^* \,.$$

(See § 1.) Consider now $A^*(f_1^* \wedge \cdots \wedge f_n^*)$. By (6.3)

$$\begin{aligned} A^*(f_1^* \wedge \cdots \wedge f_n^*) &= A^* f_1^* \wedge \cdots \wedge A^* f_n^* \\ &= \sum (a_{1,k_1} e_{k_1}^*) \wedge \cdots \wedge (a_{n,k_n} e_{k_n}^*) \end{aligned}$$

the sum being over all $k_1, \ldots, k_n$, with $1 \leq k_r \leq n$. Thus,

$$A^*(f_1^* \wedge \cdots \wedge f_n^*) = \sum a_{1,k_1} \ldots a_{n,k_n} \; e_{k_1}^* \wedge \cdots \wedge e_{k_n}^* \,.$$

If the multi-index, $k_1, \ldots, k_n$, is repeating, then $e_{k_1}^* \wedge \cdots \wedge e_{k_n}^*$ is zero, and if it's not repeating then we can write

$$k_i = \sigma(i) \quad i = 1, \ldots, n$$

for some permutation, σ, and hence we can rewrite $A^*(f_1^* \wedge \cdots \wedge f_n^*)$ as the sum over $\sigma \in S_n$ of

$$\sum a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \quad (e_1^* \wedge \cdots \wedge e_n^*)^\sigma \,.$$

But

$$(e_1^* \wedge \cdots \wedge e_n^*)^\sigma = (-1)^\sigma e_1^* \wedge \cdots \wedge e_n^*$$

so we get finally the formula

$$A^*(f_1^* \wedge \cdots \wedge f_n^*) = \det[a_{i,j}] e_1^* \wedge \cdots \wedge e_n^* \tag{6.7}$$

where

$$\det[a_{i,j}] = \sum (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \tag{6.8}$$

summed over $\sigma \in S_n$. The sum on the right is (as most of you know) the *determinant* of $[a_{i,j}]$.

Notice that if $V = W$ and $e_i = f_i$, $i = 1, \ldots, n$, then $\omega = e_1^* \wedge \cdots \wedge e_n^* = f_1^* \wedge \cdots \wedge f_n^*$, hence by (6.5) and (6.6),

$$\det(A) = \det[a_{i,j}] \,. \tag{6.9}$$

Exercises.

1. Verify the three assertions of Proposition 6.2.

2. Deduce from Proposition 6.5 a well-known fact about determinants of $n \times n$ matrices: If two columns are equal, the determinant is zero.

3. Deduce from Proposition 6.3 another well-known fact about determinants of $n \times n$ matrices: If one interchanges two columns, then one changes the sign of the determinant.

   Hint: Let $e_1, \ldots, e_n$ be a basis of V and let $B : V \to V$ be the linear mapping $Be_i = e_j$, $be_j = e_i$ and $Be_\ell = e_\ell$, $\ell \neq i, j$. What is $B^*(e_1^* \wedge \cdots \wedge e_n^*)$?

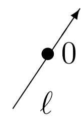
4. Deduce from Propositions 6.3 and 6.4 another well-known fact about determinants of $n \times n$ matrix. If $[b_{i,j}]$ is the inverse of $[a_{i,j}]$, its determinant is the inverse of the determinant of $[a_{i,j}]$.

5. Extract from (6.8) a well-known formula for determinants of 2×2 matrices:

$$\det \begin{bmatrix} a_{11}\,, & a_{12} \\ a_{21}\,, & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}\,.$$

7 Orientations

We recall from freshman calculus that if $\ell \subseteq \mathbb{R}^2$ is a line through the origin, then $\ell - \{0\}$ has two connected components and an *orientation* of ℓ is a choice of one of these components (as in the figure below).



More generally, if $\mathbb{L}$ is a one-dimensional vector space then $\mathbb{L} - \{0\}$ consists of two components: namely if v is an element of $\mathbb{L} - [0\}$, then these two components are

$$\mathbb{L}_1 \quad = \quad \{\lambda v, \;\; \lambda > 0\}$$

and

$$\mathbb{L}_2 \quad = \quad \{\lambda v, \;\lambda < 0\}\,.$$

An *orientation* of $\mathbb{L}$ is a choice of one of these components. Usually the component chosen is denoted $\mathbb{L}_+$, and called the *positive* component of $\mathbb{L} - \{0\}$ and the other component, $\mathbb{L}_-$, the *negative* component of $\mathbb{L} - \{0\}$.

Definition 7.1. *A vector, $v \in \mathbb{L}$, is positively oriented if v is in $\mathbb{L}_+$.*

More generally still, let V be an n-dimensional vector space. Then $\mathbb{L} = \Lambda^n(V^*)$ is one-dimensional, and we define an *orientation* of V to be an orientation of $\mathbb{L}$. One important way of assigning an orientation to V is to choose a basis, $e_1, \ldots, e_n$ of V. Then, if $e_1^*, \ldots, e_n^*$ is the dual basis, we can orient $\Lambda^n(V^*)$ by requiring that $e_1^* \wedge \cdots \wedge e_n^*$ be in the positive component of $\Lambda^n(V^*)$. If V has already been assigned an orientation we will say that the basis, $e_1, \ldots, e_n$, is *positively* oriented if the orientation we just described coincides with the given orientation.

Suppose that $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are bases of V and that

$$e_j = \sum a_{i,j,} f_i \,. \tag{7.1}$$

Then by (6.7)

$$f_1^* \wedge \cdots \wedge f_n^* = \det[a_{i,j}] e_1^* \wedge \cdots \wedge e_n^*$$

so we conclude:

Proposition 7.2. *If $e_1, \ldots, e_n$ is positively oriented, then $f_1, \ldots, f_n$ is positively oriented if and only if $\det[a_{i,j}]$ is positive.*

Corollary 7.3. *If $e_1, \dots, e_n$ is a positively oriented basis of V, the basis: $e_1, \dots, e_{i-1}$, $-e_i, e_{i+1}, \dots, e_n$ is negatively oriented.*

Now let V be a vector space of dimension $n > 1$ and W a subspace of dimension $k < n$. We will use the result above to prove the following important theorem.

Theorem 7.4. *Given orientations on V and V/W, one gets from these orientations a natural orientation on W.*

Remark. *What we mean by "natural' will be explained in the course of the proof.*

Proof. Let $r = n - k$ and let π be the projection of V onto V/W . By exercises 1 and 2 of §1 we can choose a basis $e_1, \dots, e_n$ of V such that $e_{r+1}, \dots, e_n$ is a basis of W and $\pi(e_1), \dots, \pi(e_r)$ a basis of V/W. Moreover, replacing e_1 by $-e_1$ if necessary we can assume by Corollary 7.3 that $\pi(e_1), \dots, \pi(e_r)$ is an oriented basis of V/W and replacing e_n by $-e_n$ if necessary we can assume that $e_1, \dots, e_n$ is an oriented basis of V. Now assign to W the orientation associated with the basis $e_{r+1}, \dots, e_n$.

Let's show that this assignment is "natural" (i.e., doesn't depend on our choice of $e_1, \dots, e_n$). To see this let $f_1, \dots, f_n$ be another basis of V with the properties above and let $A = [a_{i,j}]$ be the matrix (7.1) expressing the vectors $e_1, \dots, e_n$ as linear combinations of the vectors $f_1, \dots f_n$. This matrix has to have the form

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \tag{7.2}$$

where B is the $r \times r$ matrix expressing the basis vectors $\pi(e_1), \dots, \pi(e_r)$ of V/W as linear combinations of $\pi(f_1), \dots, \pi(f_r)$ and D the $k \times k$ matrix expressing the basis vectors $e_{r+1}, \dots, e_n$ of W as linear combinations of $f_{r+1}, \dots, f_n$. Thus

$$\det(A) = \det(B)\det(D)\,.$$

However, by Proposition 7.2, $\det A$ and $\det B$ are positive, so $\det D$ is positive, and hence if $e_{r+1}, \dots, e_n$ is a positively oriented basis of W so is $f_{r+1}, \dots, f_n$.

□

As a special case of this theorem suppose $\dim W = n - 1$. Then the choice of a vector $v \in V - W$ gives one a basis vector $\pi(v)$ for the one-dimensional space V/W and hence if V is oriented, the choice of v gives one a natural orientation on W.

Next let V_i, $i = 1, 2$ be an oriented n-dimensional vector space and $A : V_1 \to V_2$ a bijective linear map. A is *orientation-preserving* if, for $\omega \in \Lambda^n(V_2^*)_+$, $A^*\omega$ is in $\Lambda^n(V_+^*)_+$. For example if $V_1 = V_2$ then $A^*\omega = \det(A)\omega$ so A is orientation preserving if and only if $\det(A) > 0$. The following proposition we'll leave as an exercise.

Proposition 7.5. *Let V_i, $i = 1, 2, 3$ be oriented n-dimensional vector spaces and $A_i : V_i \to V_{i+1}$, $i = 1, 2$ bijective linear maps. Then if A_1 and A_2 are orientation preserving, so is $A_2 \circ A_1$.*

Exercises.

1. Prove Corollary 7.3.

2. Show that the argument in the proof of Theorem 7.4 can be modified to prove that if V and W are oriented then these orientations induce a natural orientation on V/W.

3. Similarly show that if W and V/W are oriented these orientations induce a natural orientation on V.

4. Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$. The *standard* orientation of $\mathbb{R}^n$ is, by definition, the orientation associated with this basis. Show that if W is the subspace of $\mathbb{R}^n$ defined by the equation, $x_1 = 0$, and $v = e_1 \notin W$ then the natural orientation of W associated with v and the standard orientation of $\mathbb{R}^n$ coincide with the orientation given by the basis vectors, $e_2, \ldots, e_n$ of W.

5. Let V be an oriented n-dimensional vector space and W an $(n-1)$-dimensional subspace. Show that if v and v' are in $V - W$ then $v' = \lambda v + w$, where w is in W and $\lambda \in \mathbb{R} - \{0\}$. Show that v and v' give rise to the same orientation of W if and only if λ is positive.

6. Prove Proposition 7.5.

7. A key step in the proof of Theorem 7.4 was the assertion that the matrix A expressing the vectors e_i as linear combinations of the vectors f_i had to have the form (7.2). Why is this the case?